Universidad del Atlántico MATUA
Facultad de Ciencias Básicas
©Programa de Matemáticas
Vol VIII | No 2, (2021), p 35-45

Research Article

Fractional Calculus: Historical Notes Apuntes Históricos del Cálculo Fraccionario

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Abstract. In this paper, we present some historical notes to Generalized Calculus, sometimes called Local Fractional Calculus, and highlight some properties and applications of these new mathematical tools.

Resumen. En este trabajo, presentamos algunos apuntes históricos al Cálculo Generalizado, a veces llamado Cálculo Fraccionario Local, y destacamos algunas propiedades y aplicaciones de estas nuevas herramientas matemáticas.

1. Introducción

Fractional calculus studies problems with derivatives and integrals of real or complex order. As a field purely mathematical, the theory of fractional calculus was proposed for the first time in the 17th century and since then many renowned scientists worked on this subject, including Euler, Laplace, Fourier, Abel, Liouville and Riemann (See [26]).

In the last 5 decades, we have witnessed the development of new operators, differential and integral, which include both fractional and generalized. The latter, in general, are defined as local derivatives and generate integral operators that may or may not be fractional. To date, the study of this area has attracted the attention of many researchers, not only in Pure Mathematics, but also in many fields of applied science. Between its own theoretical development and the multiplicity of applications, the field has grown rapidly in recent years, in such a way that a single definition of fractional derivative or integral does not exist, or at least is not unanimously accepted. in [5] the idea of a fairly complete classification of the known operators of fractional calculus is suggested and justified, but on the other hand, in [4] indicate several reasons why new operators appear in relation to the development of applications of theoretical results. These fractional calculus operators, for example, Hadamard, Riemann-Liouville (RL), Weil, Erdelyi-Kober, Katugampola and others have been defined by different mathematicians. All these operators were generated from the classical local derivatives.

The history of differential operators from Newton to Caputo, both local and global, is given in [2] (Chapter 1). Here is the definition of a local derivative with a new parameter, which has a large number of applications. More importantly, section 1.4 concludes: "Therefore, we can conclude that both the Riemann-Liouville operator and the Riemann-Liouville operator Caputo

Date: Received: 07/02/2021; Accepted: 09/10/2021.
Key words and phrases. Fractional derivatives e integral, generalized derivative, fractional calculus.

are not derivatives, and therefore they are not fractional derivatives, but fractional operators". We are of in agreement with the result [41] that says "the local fractional operator is not a fractional derivative" (page 24). As mentioned above, these tools are new and have demonstrated their potential and usefulness in solving phenomena and process modeling problems in various fields of science and technology (see [3]). Many different types of fractional operators have been proposed in the literature, here we show that several of these different notions of derivatives can be considered particular cases of our definition and, even more relevant, that it is possible to establish a direct relationship between derivatives global (classical) and local, the latter not very accepted by the mathematical community, under two arguments: its local character and compliance with the Leibniz Rule. To facilitate the understanding of the scope of our definition, we present the best known definitions of integral operators and their corresponding differential operators (for more details you can consult [6]). Without much difficulty, we can extend these definitions, for any higher order.

We assume that the reader is familiar with the classical definition of the Riemann Integral, so we will not present it.

In this work, we present an outline of the history of the Generalized Calculus, pointing out some properties and possible applications.

2. The Origins of Local Fractional Calculus

In [20] ([21] y [22]) the following fractional derivative was introduced:

Definition 2.1. Given a function $f : [0,1] \longrightarrow R$, if the limit

$$D^q f(y) = \lim_{x \to y} \frac{d^q(f(x) - f(y))}{d(x-y)^q}, \tag{2.1}$$

exists and is finite, then it is said that the local fractional derivative fraccional of f (LFD) of order q, in $x = y$, exists.

This derivative was used for the study of certain attractors of dynamic systems, which are examples of occurrence of curves and continuous surfaces, but highly irregular and non-differentiable. Frequently these graphs are fractal sets and the ordinary calculation is inadequate to characterize them. In this article the notion of local "fractional derivative " is developed, suitably modifying the concepts of fractional calculus. They also introduce the concept of local fractional integral based on the previous one, as we know it today. To understand the fractal behavior of functions, Parvate and Gangal (see [38]) introduce the derivative fractal as follows:

$$x_0 D_x^{\alpha} f(y) = \frac{d^{\alpha} f(x)}{dx^{\alpha}}(x_0) = F - \lim_{x \to x_0} \frac{f(x) - f(x_0)}{S_F^{\alpha}(x) - S_F^{\alpha}(x_0)}, \tag{2.2}$$

where the right member is the limit through the fractal F points. It is necessary to mention some ideas.

Definition 2.2. Let a be a fix and arbitrary real number . The integral of the step function $S_F^{\alpha}(x)$ of order α for a set F está is given by:

$$S_F^{\alpha}(x) = \begin{cases} \gamma^{\alpha}[F, a, x] & si \quad x \geq a \\ -\gamma^{\alpha}[F, a, x] & si \quad x < a \end{cases} \tag{2.3}$$

also we have,

Definition 2.3. The mass function $\gamma^{\alpha}[F, a, b]$ can be written as follows (See [51] and [52]):

$$\gamma^{\alpha}[F, a, b] = \lim_{\delta \longrightarrow 0} \gamma_{\delta}^{\alpha}[F, a, b] = \frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)}. \tag{2.4}$$

Meanwhile, Adda y Cresson propoused a local fractional derivative as follows:

$$x_0 D_x^\alpha f(y) = \frac{d^\alpha f(x)}{dx^\alpha}(x_0) = \lim_{x \to x_0^\sigma} D_{y,-\sigma}^\alpha \left[\sigma\left(f(x) - f(x_0)(x)\right]\right., \tag{2.5}$$

with $\sigma = \pm$ y $D_{y,-\sigma}^\alpha$ being the Riemann-Liouville fractional operactor.
Chen gave the fractal derivative notion in [7]:

$$x_0 D_x^\alpha f(y) = \frac{d^\alpha f(x)}{dx^\alpha}(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x^\alpha - x_0^\alpha}, \tag{2.6}$$

obtained from (2.5) if $x^\alpha - x_0^\alpha = (x - x_0)^\alpha$.

He proposed a new fractal derivative useful in engineering applications (see [15]):

$$x_0 D_x^\alpha f(y) = \frac{d^\alpha f(x)}{dx^\alpha}(x_0) = \lim_{\Delta x \to L_0} \frac{f(x) - f(x_0)}{K L_0^\alpha}, \tag{2.7}$$

having in account the relation

$$H^\alpha(F \cap (x, x_0)) = (x - x_0^\alpha) = \frac{K}{\Gamma(1+\alpha)} L_0^\alpha,$$

which is the Gao, Yang and Khan unified notation of a local fractional derivative (see [10]). To close this dissertation, we want to note the Yang's local fractional derivative (ver [53] and [54]), defined as follows.

$$D^\alpha f(x) = \frac{d^\alpha f(x)}{dx^\alpha}(x_0) = \lim_{x \to x_0} \frac{\Delta^\alpha \left[f(x) - f(x_0)\right]}{(x - x_0)^\alpha}, \tag{2.8}$$

where $(x - x_0)^\alpha$ is a fractal measure (see [54]) and $\Delta^\alpha \left[f(x) - f(x_0)\right] \cong \Gamma(1+\alpha)\Delta \left[f(x) - f(x_0)\right]$. A variation was introduced in [55]:

$$D^\alpha f(x) = \frac{d^\alpha f(x)}{dx^\alpha}(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{(x - x_0)^\alpha}. \tag{2.9}$$

3. Stage 2014-2019

In [19]] (see also [1]) it is defined, in a very simple way, a new local fractional derivative called "conformable " depending on a certain incremental quotient as the classical derivative

So, for a function $f : (0, \infty) \to R$ the conformable fractional derivative of order $0 < \alpha \le 1$ of f at $t > 0$ was defined by

$$T_\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \tag{3.1}$$

and the conformable fractional derivative in 0 was defined by $T_\alpha f(0) = \lim_{t \to 0} T_\alpha f(t)$.

Similarly, in a work of the same year (cf. [17]) it was defined another conformable fractional derivative.

Definition 3.1. Let $f : (0, \infty) \to R$, $t > 0$, be a function. The fractional dereivative of f, of order α con $0 < \alpha < 1$ is defined by mean the expression

$$D^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(te^{\varepsilon t^{-\alpha}}) - f(t)}{\varepsilon}$$

naturally, if $D^\alpha f(t)$ exists in a certain $(0, a)$ with $a > 0$ it is defined the fractional derivative of order α in 0 as $D^\alpha f(0) = \lim_{t \to 0} D^\alpha f(t)$.

In [16] it is introduced a new mode by generalizing the fractional derivative as follows: let $f : R \to R$ be a function and α a real number, then the fractional derivative may be considered as

$$f^{\alpha}(t) = \lim_{\varepsilon \to 0} \frac{f^{\alpha}(t+\varepsilon) - f(t)}{(t+\varepsilon)^{\alpha} - t^{\alpha}}.$$

In [40] it was defined a fractional derivative for real valued functions of several variable taking the base of the Khalil fractional derivative aforementioned.

In the work [42] it was introduced a truncated fractional derivative M for functions α-differentiables, on the base of the Mittag-Lefler of one parameter as follows:

$$(3.2) \qquad {}_iD_M^{\alpha,\beta} f(t) = \lim_{\varepsilon \to 0} \frac{f(t_i E_{\beta}(\varepsilon t^{-\alpha})) - f(t)}{\varepsilon}, \quad 0 < \alpha < 1,$$

for functions $f : (0, \infty) \to R$, $t > 0$ and $\beta > 0$. The fractional derivative in 0 is defined as the aforementioned. This fractional derivative generalize those of defined in [19], [17] y [43].

In the work of [12] it is open a new approach when it was defined a new local fractional derivative, conformable in this case, as follows.

Definition 3.2. Given a function $f : [0, +\infty) \to \mathbb{R}$, then the $N-$derivada of f, of order α is defined by

$$N_1^{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon e^{t^{-\alpha}}) - f(t)}{\varepsilon}$$

for all $t > 0$ y $\alpha \in (0, 1)$. If f id $\alpha-$differentiable in some $(0, a)$, and $\lim_{t \to 0^+} N_1^{(\alpha)} f(t)$ exists, then it is defined $N_1^{(\alpha)} f(0) = \lim_{t \to 0^+} N_1^{(\alpha)} f(t)$.

The adjective conformable may be appropriated or not, since it was initially introduced referred to $D^{\alpha} f(t)$, when $\alpha \to 1$ and satisfy $D^{\alpha} f(t) \to f'(t)$; i.e., when $\alpha \to 1$, $D^{\alpha} f(t)$ preserve the angle of the tangent line in a point of the curve, meanwhile in the above definition this is not occurs.

In [33] it was defined a more generalized fractional derivative (see also [56]).

Definition 3.3. Given a function $\psi : [0, +\infty) \to \mathbb{R}$,then the $N-$derivative of ψ, of order α is defined by

$$(3.3) \qquad N_F^{\alpha} \psi(\tau) = \lim_{\varepsilon \to 0} \frac{\psi(\tau + \varepsilon F(\tau, \alpha)) - \psi(\tau)}{\varepsilon}$$

for all $\tau > 0$, $\alpha \in (0, 1)$, being $F(\tau, \alpha)$ some function.

If ψ is $N-$differentiable on some $(0, \alpha)$, and $\lim_{\tau \to 0^+} N_F^{\alpha} \psi(\tau)$ exists, then we define $N_F^{\alpha} \psi(0) = \lim_{\tau \to 0^+} N_F^{\alpha} \psi(\tau)$. Observe that if ψ is differentiable, then $N_F^{\alpha} \psi(\tau) = F(\tau, \alpha) \psi'(\tau)$ where $\psi'(\tau)$ is the ordinary derivative.

Remark 3.4. The aforementioned generalized derivative is not fractional, but it has a desirable characteristic in applications, its dual dependency on both α and the nucleus expression. With $0 < \alpha \leq 1$ in [19] the conformable derivative is defined doing $F(t, \alpha) = t^{1-\alpha}$, meanwhile in [12] the no conformable derivative is obtained with $F(t, \alpha) = e^{t^{-\alpha}}$ (see also [32]). This generalized derivative contains as particular cases almost all the known local operators and it is useful in various applications, see [11, 13, 14, 23, 24, 27, 28, 30, 31, 34, 35, 36, 37, 44, 45]. Also, clearly, if $F \equiv 1$ it is recovered the ordinary derivative.

Remark 3.5. When (3.3) is used it must be observed that $N_F^{2\alpha} f(t) \neq N_F^{\alpha} (N_F^{\alpha} f(t))$.

Remark 3.6. From Definition 3.3 is not difficult to extend the order of the generalized derivative notion to $0 \leq n-1 < \alpha \leq n$ doing

$$N_F^\alpha h(\tau) = \lim_{\varepsilon \to 0} \frac{h^{(n-1)}(\tau + \varepsilon F(\tau, \alpha)) - h^{(n-1)}(\tau)}{\varepsilon}. \tag{3.4}$$

If $h^{(n)}$ exists on some interval $I \subseteq \mathbb{R}$, then we have $N_F^\alpha h(\tau) = F(\tau, \alpha) h^{(n)}(\tau)$, with $0 \leq n-1 < \alpha \leq n$.

Ejemplo 3.7. Some interesting cases result from choosing some special functions for the function F, such as the Mittag-Leffler function $E_{a,b}(.)$ with $Re(a), Re(b) > 0)$.

Lets see some particular cases of non conformable derivatives.

(1) Mellin-Ross function. In this case we have

$$E_t(\alpha, a) = t^\alpha E_{1,\alpha+1}(at) = t^\alpha \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(\alpha + k + 1)}$$

with $E_{1,\alpha+1}(.)$, the Mittag-Leffler function with two parameters. The we obtain $\lim_{\alpha \to 1} N_{E_t(\alpha,a)}^\alpha f(t) = f'(t) t E_{1,2}(at)$, i.e.,

$$N_{E_t(1,a)}^1 f(t) = f'(t) t \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(k+2)}.$$

(2) Robotov function.

$$R_\alpha(\beta, t) = t^\alpha \sum_{k=0}^{\infty} \frac{\beta^k t^{k(\alpha+1)}}{\Gamma(1+\alpha)(k+1)} = t^\alpha E_{\alpha+1,\alpha+1}(\beta t^{\alpha+1})$$

as before, $E_{\alpha+1,\alpha+1}(.)$, the Mittag-Leffler function of two parameters. Then we have $\lim_{\alpha \to 1} N_{R_\alpha(\beta,t)}^\alpha f(t) = f'(t) t E_{2,2}(\beta t^2)$ and

$$N_{R_1(\beta,t)}^1 f(t) = \frac{f'(t) t}{\Gamma(2)} \sum_{k=0}^{\infty} \frac{\beta^k t^{2k}}{(k+1)}.$$

(3) Let $F(t, \alpha) = E_{1,1}(t^{-\alpha})$. In this case we have, from Definition 3.3, the derivative $N_1^\alpha f(t)$ given in [12] (see also [32]).

(4) Let $F(t, \alpha) = E_{1,1}(t^{-\alpha})_1$, in this case we have $F(t, \alpha) = \frac{1}{t^\alpha}$, a new derivative with the property $\lim_{t \to \infty} N_1^\alpha f(t) = 0$, i.e., the N-derivative is zero in infinite.

(5) If we consider the expansion of E of order 1, then $E_{a,b}(t^{-\alpha}) = 1 + \frac{1}{t^\alpha}$. Therefore $\lim_{t \to \infty} N_F^\alpha f(t) = \lim_{t \to \infty} N_1^\alpha f(t) = f'(t)$, in this case we have the first order derivative.

Remark 3.8. It is easy to check but tedious, following, for example, that the general derivative fulfills properties very similar to those known in classical calculus. In addition to its most important consequences, including the chain rule, of vital importance in many applications, such as the second Liapunov method.

Following the ideas of [12] (See Theorem 3) we can easily prove the following result.

Theorem 3.9. *Let f and g be N-differentiable functions in a point $t > 0$ and $\alpha \in (0, 1]$. Then*

a) $N_F^\alpha(af + bg)(t) = a N_F^\alpha(f)(t) + b N_F^\alpha(g)(t)$.

b) $N_F^\alpha(t^p) = e^{t^{-\alpha}} p t^{t-1}, \ p \in \mathbb{R}$.

c) $N_F^\alpha(\lambda) = 0, \ \lambda \in \mathbb{R}$.

d) $N_F^\alpha(fg)(t) = f N_F^\alpha(g)(t) + g N_F^\alpha(f)(t)$.

e) $N_F^\alpha(\frac{f}{g})(t) = \frac{g N_F^\alpha(f)(t) - f N_F^\alpha(g)(t)}{g^2(t)}$.

f) If, additionally, f is differentiable, then $N_F^\alpha(f) = F(t, \alpha) f'(t)$.

Recently, in [9] it is defined a generalized fractional derivative notion from a two points of view:
1) It contains conformable an non conformable fractional derivative.
2) it is defined for any order $\alpha > 0$.
given $s \in \mathbb{R}$, we denote with $\lceil s \rceil$ integer superior part of s.

Definition 3.10. Given an interval $I \subseteq (0, \infty)$, $f : I \to \mathbb{R}$, $\alpha \in \mathbb{R}^+$ and a positive continuous function $T(t, \alpha)$, the derivative $G_T^\alpha f$ of f of order α in the point $t \in I$ is defined by

$$G_T^\alpha f(t) = \lim_{h \to 0} \frac{1}{h^{\lceil \alpha \rceil}} \sum_{k=0}^{\lceil \alpha \rceil} (-1)^k \binom{\lceil \alpha \rceil}{k} f\left(t - khT(t, \alpha)\right). \tag{3.5}$$

In 2018 it is defined a differential operator over $\mathbb{R}$ with a limit process (See [29]). For a given function p of two varible, the symbol $D_p f(t)$ defined by the limit

$$D_p f(t) = \lim_{\varepsilon \to 0} \frac{f(p(t, \varepsilon)) - f(t)}{\varepsilon},$$

whenever the limit exists, it will be called the derivative of f at t or the generalized derivative of f at t and, briefly , also it is said that f is p-differentiable at t. In the case of an closed interval, we define the p-derivative in the terminal points as the corresponding lateral derivative . From this definition , the derivative of order α is constructed for a function as follows:

$$D_p^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(p(t, \varepsilon, \alpha)) - f(t)}{\varepsilon}, \quad 0 < \alpha < 1, \tag{3.6}$$

where the ordinary derivative is recovered if $\alpha = 1$. It is clear that if f is differentiable at t, then $D_p^\alpha f(t) = p_h(t, 0, \alpha) f'(t), \quad 0 < \alpha < 1$. There is no restriction for the sign of the function p neither its partial derivative $p_h(t, 0, \alpha)$.

Therefore, from its origins, the derivative notion is "local", contrary to the global property of the integral operator, so , they are not inverse operators in strict sense. It is common to find references to instants, points, specific magnitudes and not intervals. Classical notions of fractional derivatives did not take this fact into account and and an operator that is not local was built, therefore, since their conception, classical fractional derivatives are another nature. As we have said, it is impossible to compare them, so Tarasov's statements should be reformulated as follows: *"There is no locality, there is no differential operator"*.

Noncompliance with the product rule was always considered to be a feature of fractional (global) derivatives. However, a new local derivative can be constructed that violates Leibniz's Rule, so violation of this rule cannot be a necessary condition for a given operator to be a fractional derivative, back to (3.3). It is clear that the violation of this rule does not depend (at least not only) on the incremental quotient, but on a factor that we can add to the incremented function, from which we would obtain the non-symmetry of the product rule.

Having in account [57] we can write form (3.3) the following derivative ($\alpha + \beta = 1$):

$$DH_\beta^\alpha f(t) := \lim_{\varepsilon \to 0} \frac{H(\varepsilon, \beta) f(t + \varepsilon F(t, \alpha)) - f(t)}{\varepsilon} \tag{3.7}$$

with $H(\varepsilon, \beta) \to k$ if $\varepsilon \to 0$. In the case of $k \equiv 1$, we can consider two simple cases:

I) $H(\varepsilon, \beta) = 1 + \varepsilon\beta$ as in [57] and so

$$DL_\beta^\alpha f(t) := \lim_{\varepsilon \to 0} \frac{(1 + \varepsilon\beta) f(t + \varepsilon F(t, \alpha)) - f(t)}{\varepsilon}.$$

If $F(t, \alpha) = e^{t^{-\alpha}}$, i.e., a generalization of the local fractional derivative presented in example 4 above. In this case we have:

(3.8)
$$NL_2^\alpha f(t) := \lim_{\varepsilon\to 0}\frac{(1+\varepsilon\beta)f(t+\varepsilon e^{t^{-\alpha}})-f(t)}{\varepsilon}.$$

II) $H(\varepsilon,\beta)=1+\varepsilon\beta^r$, $r>0$, so we have

$$DP_\beta^\alpha f(t) := \lim_{\varepsilon\to 0}\frac{(1+\varepsilon\beta^r)f(t+\varepsilon F(t,\alpha))-f(t)}{\varepsilon}.$$

With reference to the N-derivative in [12], we obtain:

(3.9)
$$NP_2^\alpha f(t) := \lim_{\varepsilon\to 0}\frac{(1+\varepsilon\beta^r)f(t+\varepsilon e^{t^{-\alpha}})-f(t)}{\varepsilon}.$$

If $k\neq 1$, when $e^x = 1+x+\frac{x^2}{2!}+...$ we can take (as a first possibility):

III) $H(\varepsilon,\beta)=E_{1,1}(\varepsilon\beta)$ and then we have

$$DE_\beta^\alpha f(t) := \lim_{\varepsilon\to 0}\frac{E_{1,1}(\varepsilon\beta)f(t+\varepsilon F(t,\alpha))-f(t)}{\varepsilon},$$

and with our N-derivative in [12], it follows:

(3.10)
$$NE_\beta^\alpha f(t) := \lim_{\varepsilon\to 0}\frac{E_{1,1}(\varepsilon\beta)f(t+\varepsilon e^{t^{-\alpha}})-f(t)}{\varepsilon}.$$

From(3.7) we can easily obtain the following conditions:

(1) it is a local derivative, i. e. , it is defined in point.
(2) They are derivatives,in the strict sense.
(3) It doesn't follow the Leibniz rule, the for (3.8) we have (similar steps for (3.9) and (3.10)):

$$NL_2^\alpha\left[f(t)g(t)\right] = \left(N_2^\alpha f(t)\right)g(t) + f(t)\left(N_F^\alpha g(t)\right),$$

Also for (3.8) we have (again for (3.9) and (3.10)):

(4) if $\alpha=0$, $\beta=1$ then $N_2^\alpha f(t) = N_F^0 f(t)+f(t)=(1+e)f(t)$.
(5) if $\alpha=1$, $\beta=0$ then

$$\begin{aligned} N_2^1 f(t) = N_{e^{t^{-1}}}^1 f(t) &= \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon e^{t^{-1}})-f(t)}{\varepsilon}\\ = \quad e^{t^{-1}}&\left[\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon e^{t^{-1}})-f(t)}{\varepsilon e^{t^{-1}}}\right] = e^{t^{-1}}f'(t),\end{aligned}$$

if f is differentiable.

(6) if the limit exists in (3.10) then we have

(3.11)
$$NL_\beta^\alpha f(t) = N_F^\alpha f(t)+\beta f'(t).$$

(7) unfortunately, we lost the chain rule although remain valid for the N-derivative (see [12]), the for NL_β^α we obtain:

$$NL_\beta^\alpha\left[f(g(t))\right] = N_F^\alpha f(g(t))+\beta f(g(t)).$$

(8) From (3.11) it is deduced that

$$\begin{aligned} \lim_{t\to\infty} NL^{\alpha}_{\beta} f(t) &= \lim_{t\to\infty} N^{\alpha}_{F} f(t) + \lim_{t\to\infty} \beta f'(t) \\ &= f'(t) + \beta f(\infty). \end{aligned}$$

We can describe the following: if the term $\beta f(\infty)$ exists, then $N^{\alpha}_{\beta} f(t)$ is an only translate of the derivative of the function when $t \to \infty$, but it not affects the qualitative behavior of the ordinary derivative, this is of vital importance in the study of the asymptotic properties of the solutions of fractional differential equations with NL^{α}_{β}. Lamentably, the non-existence of the limit of the function at infinity makes impossible the qualitative study of these fractional differential equations.

(9) Again, from (3.7), it is clear that the function $H(\varepsilon, \beta)$ can be generalized, although that will extraordinarily complicate the calculations. Of course, this does not close the discussion about what terms can be "added" to the augmented function giving local fractional derivatives that violate the Leibniz Rule, which would hold the locality, as a historical inheritance of the derivative, and that would violate Leibniz, as a "necessary" condition for there to be a fractional derivative.

4. Present times

In [25] it was defined a new generalized derivative, with an interesting characteristic that will be exposed later. So we have:

Definition 4.1. Let $f : [0, +\infty) \to \mathbb{R}$, $\alpha_i \in (0, 1]$ para $i = 1, 2, ..., n$ and $F(t, \alpha)$ some absolutely continuous function over $I \times (0, 1]$. Then, the $N-$ multindex derivative of f, of order $\alpha_1 + \alpha_2 + ... + \alpha_n$ is defined by

$$N_F^{\langle \alpha_1, \alpha_2, \dots, \alpha_n \rangle} f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon F(t, \alpha_1, \alpha_2, ..., \alpha_n)) - f(t)}{\varepsilon}, \quad t > 0. \tag{4.1}$$

For the kernel function, we consider the following:

- $F(t, \alpha_1, \alpha_2, ..., \alpha_n) \neq 0, \forall t \in \mathbb{R}^+$
- $F(t, \alpha_1, ..., \alpha_i, ..., \alpha_n) \neq F(t, \alpha_1, ..., \alpha_j, ..., \alpha_n)$, $\forall i \neq j$, with $i, j = 1, 2, ..., n$

If f is generalized $N-$differentiable for $0 < \alpha_1, \alpha_2, ..., \alpha_n \leq 1$, y $\lim_{t\to 0^+} N_F^{\langle \alpha_1, \alpha_2, \dots, \alpha_n \rangle} f(t)$ exists, then it is defined

$$N_F^{\langle \alpha_1, \alpha_2, \dots, \alpha_n \rangle} f(0) = \lim_{t\to 0^+} N_F^{\langle \alpha_1, \alpha_2, \dots, \alpha_n \rangle} f(t).$$

Remark 4.2. In the event that it is fulfilled

$$\lim_{(\alpha_1, \alpha_2, \dots, \alpha_n) \to (1, 1, \dots, 1)} F(t, \alpha_1, \alpha_2, ..., \alpha_n) = 1 \tag{4.2}$$

then it will be said that this derivative is the conformable generalized multindexed $N-$derivative, in other case it will be non conformable.

Remark 4.3. Taking $n = 1$, if $F(t, \alpha) = t^{1-\alpha}$ then from Definition 4.1 we obtain the conformable Khalil et. al. derivative [19]. If $F(t, \alpha) = E_{1,1}(t^{-\alpha})$ then from the aforementioned Definition give us the non conformable derivative in [12, 32]. Other choices of the kernel function head towards to different and known local derivatives. (See [1, 18]).

Remark 4.4. If $n = 2$, then we obtain "$\alpha_1\alpha_2$ derivada " and with $\varepsilon F(t, \alpha_1, \alpha_2) = h(t)^{\alpha_2} E_{1,1}[\varepsilon k^{1-\alpha_1}(t)]$, then we have the "$\alpha, \beta$" conformable derivative of f, of order $\alpha + \beta$ (see [39]).

Remark 4.5. The important thing about the Definition 4.1 is that it shows its generality and amplitude, and that it can include generalized derivatives of a new type (not reported in the literature)

The detail that we want to highlight about this derivative and that is very characteristic, is the fact that in a derivative of order α there are successive derivatives "within said order" and not as in the entire case, that it must be fulfilled that the successive derivatives are of higher order than the first.

5. Some open problems

Regarding what has already been established regarding generalized conformable and non-conformable derivatives, there are currently some open problems, or at least not found in the literature, such as:

- Representation of the fourier transform
- Engineering applications using generalized Fourier transforms by means of conformable and non-conformable integral and/or differential operators
- Development of partial fractional differential equations for the different conformable and non-conformable fractional derivatives
- Comparative studies between these derivatives and their applications
- Studies of these derivatives in generalized vector spaces
- In the field of integral inequalities there are some works that relate this topic with the Riemman-Liouville fractional integral and another much more generalized one such as that of R. K. Raina [8, 46, 47, 48, 49, 50]. Since this area is very large, much remains to be done by applying conformable and non-conformable fractional derivatives.

6. Conclusiones

In this informative article, a historical summary has been presented, starting with the origins of the fractal derivative and then with a stage between 2014 and the present, regarding generalized fractional derivatives, conformable and not conformable. Additionally, its origins, characteristics, properties and applications have been established, as well as the way in which one and the other are deduced in some cases.

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