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Research Article

Asymptotic estimates for a nonlinear wave problem

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Abstract. We study the asymptotic solutions for a nonlinear damped wave problem showing the existence and uniqueness of solutions by contraction

$$\begin{cases} v_{tt} + 2\kappa v_t - \eta v_{xx} = -\delta\,|v|^{\rho}\,v + g(t,x), \\ v\,(0,x) = \gamma\,(x)\,, \;\; v_t\,(0,x) = \chi\,(x)\,, \;\; t>0, \;\; x \in \Omega, \end{cases}$$

where $\kappa, \eta, \delta, \rho > 0$, $g(t,x)$ is the external force. The initial data are periodic with respect to the spatial variable x. In this research we demonstrate that if the initial data $\gamma \in \mathbf{H}^1$ and $\chi \in \mathbf{L}^2$ showing that there is a single solution

$$v(t,x) \in C([0,+\infty) : \mathbf{H}^1) \text{ for all } t > 0.$$

In addition we have the following decay estimates

$$||v(t)||_{\mathbf{L}^\infty} \leq K\langle t\rangle^{\frac{1}{\rho}}, \;\; ||\partial_t v(t)||_{\mathbf{L}^\infty} \leq K\langle t\rangle^{-\frac{1}{\rho}-\frac{3}{2}}.$$

1. Introduction

In this work, we will study the nonlinear cushioned wave equation problem with periodic initial data given by

$$\begin{cases} v_{tt} + 2\kappa v_t - \eta v_{xx} = -\delta|v|^{\rho}v + g(t,x), \\ v(0,x) = \gamma(x), \;\; v_t(0,x) = \chi(x), \;\; x \in \mathbb{R}, \;\; t>0, \end{cases} \tag{1.1}$$

where $\kappa, \eta, \delta, \rho > 0$.

The external force $g(t,x)$ and the initial data $\gamma(x)$ a nd $\chi(x)$ are periodic in nature with respect to x (spatial variable), i.e. $g(t,2\pi+x) = g(t,x)$, $\gamma(2\pi+x) = \gamma(x)$, $\chi(2\pi+x) = \chi(x)$, for all $x \in \mathbb{R}$, $t > 0$.

This problem complements the study done in [8], where $g(t,x)$ is identically null. The present paper is relevant, since for the applications where the presence of external forces is natural that influence and determine the behavior asymptotic solutions. Additionally, the presence of an external force creates difficulties to extend the solutions to global times. These difficulties would

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entail to obtain some inequalities that may be useful for specialists in this field of mathematics. On the other hand, there are many physical interpretations that these external forces have, for example, we can measure, according to the magnitude of this force, the variation of the amplitude of the wave and how it warps over time.

Below, we survey the state of art of this kind of problem. In [4], the asymptotic behavior of solutions of a nonlinear periodic problem of Sobolev type is studied, where decay estimates are obtained for a nonlinear equation of evolution.

In [9], several non-linear evolution equations using computational techniques are studied. Also, in [9] sub-critical powers are studied, critical and super critical with nonlinearities of the Sobolev type. In [5] the study of asymptotic behavior for a non-linear cubic equation type Schrödinger. In [11] and [1], the study of an initial-value problem for Burgers equation with variable coefficients was done. In [8], it is established asymptotic results for the nonlinear damped wave equation,the sub-critical $(0 < \rho < 2)$, critical $(\rho = 2)$ and super critical $(\rho > 0)$ cases, which motivates our research, moreover with this methodology our future results generalizes those obtained for [8]. In [4] dispersive relationships and properties of the Sobolev periodical spaces were studied. In [13] the associated semi-groups were studied, this work will help us to understanding what kind of solution satisfies our equation, in the sense of convergence, extension of the local solution and verification of initial conditions. In [10] different non-linear evolution equations are studied with one argument similar approach by u in our work.

It is important to mention that the asymptotic behavior of solutions to the periodic problem for the nonlinear wave equation (1.1) has not been studied with the presence of an external force. This is not an expected conclusion with the given conditions that the external force generates technical complications, which are explained in detail in our publication. In summary, our working methodology is inspired by [8] and [2], where in the latter work was studied the different behavior of the IVP for different value of ρ.
Also very interested in understanding the dynamics of solutiuons, in particular studying the Koopman operators associated with the equation that have recently become of great interest in current mathematical research.
Other applications of (1.1) are in the theory of elasticity of materials and in quantum fields, see for instance [12], [7].

2. Linear Problem

As the main focus of our paper is to make asymptotic estimation of the solutions of (1.1), initially, we will solve the associated linear problem. For this, the one-dimensional Fourier transformation is used

$$\begin{cases} v_{tt} + 2\kappa v_t - \eta v_{xx} = g(t,x), \\ v(0,x) = \gamma(x), v_t(0,x) = \chi(x), x \in \mathbb{R}, t > 0, \end{cases} \tag{2.1}$$

where g is a know function of class $C^\infty((0,\infty) \times \mathbb{R} : \mathbb{R})$, periodic with respect to x (spatial variable). The initial data $\gamma, \chi, \in C^\infty_{per}(\mathbb{R} : \mathbb{R})$.

Green operator is defined by

$$\mathcal{F}(t)\chi = e^{-\kappa t} \sum_{n=-\infty}^{\infty} \frac{\widehat{\chi_n} e^{inx} \sin(t\sqrt{\eta n^2 - \kappa^2})}{\sqrt{\eta n^2 - \kappa^2}},$$

therefore, the solution of the linear problem can be written using the Duhamel's formula

$$v(t) = \tilde{\mathcal{F}}(t)\gamma + \mathcal{F}(t)\chi + \int_0^t \mathcal{F}(t-\tau) g(t) d\tau,$$

where $\tilde{\mathcal{F}} = (2\kappa + \partial_t)\mathcal{F}$. Next, we will make some estimates for the solutions of the integral equation associated with the equation (1.1) in the Sobolev's spaces $\mathbf{H}^s$ and $\dot{\mathbf{H}}^s$, defined by

$$\mathbf{H}^s = \left\{ \gamma \in \mathcal{P}' : ||\gamma||^2 = \sum_{n=-\infty}^{n=\infty} \langle n \rangle^{2s} |\widehat{\gamma}_n|^2 < \infty \right\},$$

$$\dot{\mathbf{H}}^s = \{\gamma \in \mathbf{H}^s : \hat{\gamma}_0 = 0\}.$$

$\dot{\mathbf{H}}^s$ it is know as the homogeneous Sobolev's space $\mathbf{H}^s$. For $n \in \mathbb{Z}$, let us denote $\langle n \rangle = \sqrt{1+n^2}$ and $\hat{\gamma}_n = \frac{1}{2\pi}\int_\Omega e^{-inx}\gamma(x)dx$ the coefficients of the Fourier series with the variable space $x \in [-\pi, \pi]$.

We will denote by $\mathcal{P}$ the space of the infinitely differentiable and 2π-periodic functions. $\mathcal{P}'$ is the dual topological of $\mathcal{P}$. This space is built using linear transformations of lines of $\mathcal{P}$ into $\mathbb{C}$.These are called periodic distributions, for more details see [4, 5]. In the continuous case $\langle x \rangle = \sqrt{1+x^2}$, with $x \in \mathbb{R}$ is obtained the solution of problem (2.1).

We will define the following operator

$$\mathcal{A}w(t) = \tilde{\mathcal{F}}(t)\gamma + \mathcal{F}(t)\chi + \int_0^t \mathcal{F}(t-\tau)g(\tau)d\tau + \int_0^t \mathcal{F}(t-\tau)\mathcal{N}(\tau)d\tau,$$

with $C([0,T] : \mathbf{H}^s)$ as domain and codomain.

This transformations is a contraction in a complete metric space, and thus has a fixed point which guarantees solutions for small times of the integral equation associated with the nonlinear problem (1.1). The solution given by the integral equation will be called moderate solution of the equation (1.1). In this investigation we will demonstrate that if the initial data $\gamma \in \mathbf{H}^1$ and $\chi \in \mathbf{L}^2, g \in \mathbf{H}^s$, with $s > \frac{1}{2}$, and g satisfy the inequality $||g||_{\mathbf{H}^s} \leq K\langle t \rangle^{\frac{1}{\rho}-\frac{3}{2}},\ t \geq 0$, then there is one mild solution

$$v(t,x) \in C([0,\infty) : \mathbf{H}^1)$$

of the problem with periodic initial data. This solution has also estimation of decay as follows

$$||v(t)||_{\mathbf{L}^\infty} \leq K\langle t \rangle^{-\frac{1}{\rho}},\ \ ||\partial_t v(t)||_{\mathbf{L}^\infty} \leq K\langle t \rangle^{-\frac{1}{\rho}-\frac{1}{2}},$$

for all $t > 0$.Moreover, under some initial conditions, we will find asymptotic formulas for the solution.

Remark 2.1. The previous estimate on g is the most optimal since we want that its decay is faster than those of the solution of the problem (1.1) with $g \equiv 0$. The extension of this study to external force presents considerable additional difficulties that can be evidenced in the local and global solutions, respectively.

We can also reach this conclusion by comparing with the linear case. There are various points used in the hypothesis about g, mainly in the existence global solutions to guarantee estimation (2.1) and asymptotic solution points (4.8) and (5.3).

Recently, considerable attention has been given to the initial value problem for hyperbolic equations with different types of nonlinearities. In our particularly case, our equation is reduced to several know equations of mathematical physics, i.e. the Poisson equation when $\kappa = 0,\ \eta < 0$.

In our investigation we prove that if the initial data $\gamma \in \mathbf{H}^1,\ \chi \in \mathbf{L}^2$ and $g \in \mathbf{H}^s$, with $s > \frac{1}{2}$, along with a condition on $g \in \mathbf{H}^s$ and that is enough to obtain the local solution of our wave problem. In order for the global solution and asymptotic formulas that we arrived at in the main theorems of our article to be obtained, it is necessary that g satisfy the following inequality:

$$||g||_{\mathbf{H}^s} \leq K\langle t\rangle^{-\frac{1}{\rho}-\frac{3}{2}}$$

Then, there is only one solution $v(t,x) \in C([0,\infty) : \mathbf{H}^1)$ of the periodic problem, which has estimates of decay

$$||v(t)||_{\mathbf{L}^\infty} \leq K\langle t\rangle^{\frac{1}{\rho}}, \ ||\partial_t v(t)||_{\mathbf{L}^\infty} \leq K\langle t\rangle^{-\frac{1}{\rho}-\frac{3}{2}},$$

for all $t > 0$.

We will present the following lemma which will help us for our estimate:

Lemma 2.2. *Let $\gamma \in \mathbf{H}^s$, $\chi \in \mathbf{H}^{s-1}$ with $s \geq 0$. Then the following estimates are true all $t > 0$ and $\mu = Re(\kappa - \sqrt{\kappa^2 - \eta}) > 0$.*

$$\begin{aligned}
\left\|\tilde{\mathcal{F}}(t)\gamma\right\|_{\mathbf{H}^s} &\leq K\left\|\gamma\right\|_{\mathbf{H}^s},\\
\left\|\tilde{\mathcal{F}}(t)\gamma\right\|_{\dot{\mathbf{H}}^s} &\leq Ke^{-\mu t}\left\|\gamma\right\|_{\dot{\mathbf{H}}^s},\\
\left\|\mathcal{F}(t)\chi\right\|_{\mathbf{H}^s} &\leq K\left\|\chi\right\|_{\mathbf{H}^{s-1}},\\
\left\|\mathcal{F}(t)\chi\right\|_{\dot{\mathbf{H}}^s} &\leq Ke^{-\mu t}\left\|\chi\right\|_{\dot{\mathbf{H}}^{s-1}},\\
\left\|\int_0^t \tilde{\mathcal{F}}(t-\tau)g(\tau)d\tau\right\|_{\dot{\mathbf{H}}^s} &\leq K\int_0^t \left\|g(\tau)\right\|_{\dot{\mathbf{H}}^{s-1}}d\tau,\\
\left\|\int_0^t \mathcal{F}(t-\tau)g(\tau)d\tau\right\|_{\dot{\mathbf{H}}^s} &\leq Ke^{-\mu t}\int_0^t \left\|g(\tau)\right\|_{\dot{\mathbf{H}}^{s-1}}d\tau.
\end{aligned}$$

Proof. See [2]. □

3. **Local existence of a moderate solution**

Now, the local existence of moderate solution of the equation (1.1) will be shown, via contraction mapping.

Lemma 3.1. *Let the initial data $\gamma \in \mathbf{H}^s, \chi \in \mathbf{H}^{s-1}, g \in \mathbf{H}^s$ with $1 \geq s > \frac{1}{2}$, $\rho > 0$. Then for some time $T > 0$ there is only one solution*

$$v(t,x) \in C([0,T] : \mathbf{H}^s),$$

of the periodic problem (1.1).

Proof. Let us denote by $\mathcal{N}(v) = -\delta|v|^\rho v$. Under the Green operator $\mathcal{F}(t)$ of the periodic problem (1.2) we will write the nonlinear periodic problem in the form of an equation integral

$$v(t,x) = \widetilde{\mathcal{F}}(t)\gamma + \mathcal{F}(t)\chi + \int_0^t \mathcal{F}(t-\tau)\mathcal{N}(v)(\tau)d\tau.$$

We will solve the integral equation using the principle of contraction.For this, it is used the transformation

$$\mathcal{A}w(t) = \widetilde{\mathcal{F}}(t)\gamma + \mathcal{F}(t)\chi + \int_0^t \mathcal{F}(t-\tau)\mathcal{N}(w)(\tau)d\tau,$$

in space $\overline{B}(0,R) \subset C([0,T] : H^s)$ where $R = 2K(||\gamma||_{\mathbf{H}^s} + ||\chi||_{\mathbf{H}^{s-1}})$, we must estimates $T > 0$, because of the inequality:

$$|\mathcal{N}(v) - \mathcal{N}(w)| \leq K||v|^\rho + |w|^\rho||v - w|,$$

We have

$$||\mathcal{N}(v) - \mathcal{N}(w)||_{\mathbf{H}^{s-1}} \leq K\left(||v||^\rho_{\mathbf{H}^s} + ||w||^\rho_{\mathbf{H}^s}\right)||v - w||_{\mathbf{H}^s}, \ \ s > \frac{1}{2}.$$

And thanks to lemma 3.1, we have too:

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|\mathcal{A}w(t)\|_{\mathbf{H}^s}\\
&\le \sup_{t\in[0,T]} \Big(\|\widetilde{\mathcal{F}}(t)\gamma\|_{\mathbf{H}^s} + \|\mathcal{F}(t)\chi\|_{\mathbf{H}^s}\\
&+ \left\| \int_0^t \mathcal{F}(t-\tau)\mathcal{N}(w)(\tau)d\tau \right\|_{\mathbf{H}^s} + \left\| \int_0^t \mathcal{F}(t-\tau)g(\tau)d\tau \right\|_{\mathbf{H}^s} \Big)\\
&\le K\|\gamma\|_{\mathbf{H}^s} + K\|\chi\|_{\mathbf{H}^{s-1}} + KT \sup_{t\in[0,T]} \|\mathcal{N}(w)(t)\|_{\mathbf{H}^{s-1}} + KT \sup_{t\in[0,T]} \|g\|_{\mathbf{H}^{s-1}}\\
&\le K\|\gamma\|_{\mathbf{H}^s} + K\|\chi\|_{\mathbf{H}^{s-1}} + KT \sup_{t\in[0,T]} \|w\|^{\rho+1}_{\mathbf{H}^s} + KT \sup_{t\in[0,T]} \|g\|_{\mathbf{H}^{s-1}}.
\end{aligned}
$$

Then there is a sufficiently small T that depends on the norm of the initial data $||\gamma||_{\mathbf{H}^s}+||\chi||_{\mathbf{H}^{s-1}}$ such that

$$\sup_{t\in[0,T]} \|\mathcal{A}w(t)\|_{\mathbf{H}^s} \le 2K(||\gamma||_{\mathbf{H}^s} + ||\chi||_{\mathbf{H}^{s-1}})$$

Let us now estimate the difference

$$
\begin{aligned}
\sup_{t\in[0,T]} \|\mathcal{A}v(t) - \mathcal{A}w(t)\|_{\mathbf{H}^s} &\le \sup_{t\in[0,T]} \|\mathcal{F}(t-\tau)\left(\mathcal{N}(v)(\tau) - \mathcal{N}(w)(\tau)\right)\|_{\mathbf{H}^s}\\
&\le KT \sup_{t\in[0,T]} \|v(t) - w(t)\|_{\mathbf{H}^s}(||v||^{\rho}_{\mathbf{H}^s} + ||w||^{\rho}_{\mathbf{H}^s})\\
&\le \frac{1}{2} \sup_{t\in[0,T]} \|v(t) - w(t)\|_{\mathbf{H}^s}
\end{aligned}
$$

where $T > 0$ is small enough.

Therefore the transformation $\mathcal{A}$ is a contraction in the closed ball of radius $2K(||\gamma||_{\mathbf{H}^s} + ||\chi||_{\mathbf{H}^{s-1}}$ in space $C([0,T] : \mathbf{H}^s)$. Then there is a single solution $u(t,x) \in C([0,T] : \mathbf{H}^s)$ of the periodic problem (1.1).

□

4. Global existence of moderate solution

Lemma 4.1. *Let us assume the initial data $\gamma \in \mathbf{H}^s, \chi \in \mathbf{L}^2, g \in \mathbf{H}^s$ and the inequality $||g||_{\mathbf{H}^s} \le C\langle t\rangle^{-\frac{1}{\rho}-\frac{3}{2}}$ with $s > \frac{1}{2}$ for $t \ge 0$. Then there is only one solution in global time $u(t,x) \in C([0,\infty) : \mathbf{H}^1)$ of the periodic problem (1.1). Moreover, for every $\epsilon > 0$ there is a time T such that*

$$||v(t)||_{\mathbf{H}^s} + ||v_t(t)||_{\mathbf{L}^2} \le \varepsilon \text{ for all } t \ge T.$$

Proof. We will use energy type estimates. Let u be the solution built in (Lemma 3.1), we multiply the equation (1.1) by $2u_t + \kappa u$. So integrating over Ω, we will have

$$
\begin{aligned}
\int_\Omega g(t)(2v_t + \kappa v)dx = \int_\Omega &(2v_t v_{tt} + 4\kappa v_t^2 - 2\eta v_t v_{xx} + 2\delta|v|^\rho v_t v\\
&+ \kappa v_{tt} v + 2\kappa^2 v_t v - \eta\kappa v_{xx} + \delta\kappa|v|^{\rho+2})dx.
\end{aligned}
$$

With the boundary conditions and taking integrating by parts, we will have

$$\int_\Omega f(t,x)(2v_t + \kappa v)dx = \frac{dE}{dt} + H,$$

where

$$E(t) = \int_\Omega \left(v_t^2 + \kappa v_t + \kappa^2 v^2 + \frac{2\delta}{\rho+2}|v|^{\rho+2} \right) dx,$$
$$H(t) = \int_\Omega \left(3\kappa v_t^2 + \eta\kappa v_x^2 + \delta\kappa |v|^{\rho+2} \right) dx.$$

Let us note that $H(t) \geq 0$ for $t \in [0,T]$. On the other hand,

$$H(t) - \int_\Omega g(t,x)(2v_t + \kappa v)dx \geq 0, \ \ \forall t \in [0,T], \tag{4.1}$$

and we will have to $\frac{dE}{dt} \leq 0$ for $t \in [0,T]$. Inequality (4.1) is easy to observe by Sobolev immersion theorem and hypothesis $||g||_{\mathbf{H}^s} \leq K\langle t\rangle^{-\frac{1}{\rho}-\frac{3}{2}}$.

For more clarity let us argue in detail the inequality (4.1). Consider the estimates for

$$H(t) - \int_\Omega g(t,x)(2v_t + \kappa v)dx. \tag{4.2}$$

With the Cauchy-Schwarz inequality it will be estimated $|\int_\Omega f(t)(2v_t+\kappa v)dx|$.

$$\begin{aligned}\left| \int_\Omega g(t)(2v_t + \kappa v)dx \right| &\leq ||g||_{\mathbf{L}^2} ||2v_t + \kappa v||_{\mathbf{L}^2} \\ &\leq K||g||_{\mathbf{L}^2} \left(||v_t||_{\mathbf{L}^2} + ||v||_{\mathbf{L}^2} \right) \\ &\leq K||g||_{\mathbf{L}^2} \sqrt{H}.\end{aligned}$$

As $||v_t||^2_{\mathbf{L}^2} \leq KH$ and by Sobolev immersion theorem $||v||^2_{\mathbf{L}^2} \leq KH$ and using the inequality

$$ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2,$$

so

$$\begin{aligned}\left| \int_\Omega g(t)(2v_t + \kappa v)dx \right| &\leq K||g||_{\mathbf{L}^2} \sqrt{H} \\ &\leq \frac{1}{2} H + K||g||^2_{\mathbf{L}^2}.\end{aligned}$$

Then we will give the following estimate

$$\begin{aligned} H - \int_\Omega g(t)(2v_t + \kappa v)dx &\geq H - \frac{1}{2}H + K||g||^2_{\mathbf{L}^2} \\ &\geq \frac{1}{2}H + K||g||^2_{\mathbf{L}^2} \\ &\geq \frac{1}{2}H + K\langle t\rangle^{\frac{2}{\rho}-3}.\end{aligned}$$

then

$$\frac{dE}{dt} = -H + \int_\Omega g(t,x)(2v_t + \kappa v)dx \leq -\frac{1}{2} + K\langle t\rangle^{\frac{2}{\rho}-3}.$$

Let us note that can be taken $\varepsilon > 0$, which depends only on the parameters κ, η, δ and ρ such that $H - \varepsilon E \geq 0$. Then

$$E(t) \leq E(0)e^{-\frac{\varepsilon}{2}t} + K \int_0^t e^{-\frac{\varepsilon}{2}(t-\tau)\langle\tau\rangle^{-\frac{2}{\rho}-3}} d\tau.$$

On the other hand

$$\int_0^t e^{-\frac{\varepsilon}{2}(t-\tau)\langle\tau\rangle^{-\frac{2}{\rho}-3}}d\tau = \int_0^{\frac{t}{2}} e^{-\frac{\varepsilon}{2}(t-\tau)}\langle\tau\rangle^{-\frac{2}{\rho}-3}d\tau + \int_{\frac{t}{2}}^t e^{-\frac{\varepsilon}{2}(t-\tau)}\langle\tau\rangle^{-\frac{2}{\rho}-3}d\tau$$

$$\leq e^{-\frac{\varepsilon}{2}\frac{t}{2}}\int_0^{\frac{t}{2}}\langle\tau\rangle^{-\frac{2}{\rho}-3}d\tau + \langle\frac{t}{2}\rangle^{-\frac{2}{\rho}-3}\quad\int_{\frac{t}{2}}^t e^{-\frac{\varepsilon}{2}(t-\tau)}d\tau \to 0,$$

for $t\to\infty$. Therefore the norm of the solution

$$||v(t)||_{\mathbf{H}^1} + ||v_t(t)||_{\mathbf{L}^2} \to 0\text{for } t\to\infty.$$

□

By Lemma 4.1 we can reduce the asymptotic study to the case of initial data of small norm.

Remark 4.2. In the next theorems, 5.1 and 6.1, we will use slogans 2.4 and 2.5, respectively, of [5] to arrive at estimates of decay (5.1) and asymptotic (4.1).

5. Decay estimates

Theorem 5.1. *Suppose the initial data* $\gamma\in\mathbf{H}^1, \chi\in\mathbf{L}^2$ *and* $g\in\mathbf{H}^s$ *with hypotheses qualitative* $||g||_{\mathbf{H}^s}\leq K\langle t\rangle^{-\frac{1}{\rho}-\frac{3}{2}}$ *with* $s>\frac{1}{2}$*. Then there is only one solution*

$$v(t,x)\in C([0,\infty):\mathbf{H}^1),$$

of the periodic problems (1.1), which satisfies the following decay estimates

$$||v(t)||_{\mathbf{L}^\infty}\leq K\langle t\rangle^{\frac{1}{\rho}},\;\; ||\partial_t v(t)||_{\mathbf{L}^\infty}\leq K\langle t\rangle^{-\frac{1}{\rho}-\frac{3}{2}}. \tag{5.1}$$

Proof. The global existence of a single solution $v(t,x)\in C([0,\infty):\mathbf{H}^1)$ of periodic nonlinear problem (1.1) comes from lemma 4.1 after a time T, so we will have

$$||v(t)||_{\mathbf{H}^1} + ||v_t(t)||_{\mathbf{L}^2}\leq\varepsilon, \tag{5.2}$$

for all $t\geq T$.

To test the asymptotic relation of the solution, consider the periodic problem (1.1) with the initial time T for the change $t'=t-T$, , we rewrite the periodic problem (1.1) with the initial time at the origin and small initial data $\gamma\in\mathbf{H}^1,\chi\in\mathbf{L}^2$. Let us denote the average value of the solution for

$$h(t)=\widehat{v}_0(t)=\frac{1}{2\pi}\int_\Omega v(t,x)dx,$$
$$v(t,x)=h(t)+r(t,x),$$

for $h(t)$ we will get form the equation

$$h''(t)+2\kappa h'(t)=\widehat{N}_0(h+r)+f_0(t), \tag{5.3}$$
$$\widehat{N}_0(v)=\frac{-\delta}{2\pi}\int_\Omega |v(t,x)|^\rho v(t,x)dx,$$
$$g_0(t)=\frac{1}{2\pi}\int_\Omega g(t,x)dx.$$

For the function $r(t,x)$ we will have the following integral equation

$$\begin{aligned} r(t,x) &= v(t,x)-h(t,x)\\ &= \mathcal{F}(t)(\gamma-\gamma_0)+\mathcal{F}(t)(\chi-\chi_0)+\int_0^t\mathcal{F}(t-\tau)\left(\mathcal{N}(v)(\tau)-\mathcal{N}_0(\tau)\right)d\tau\\ &+\int_0^t\mathcal{F}(t-\tau)\left(g(\tau)-g_0(\tau)\right)d\tau.\end{aligned}\tag{5.4}$$

By (5.2) we will have the inequality

$$|h(t) + |h'(t)| \leq \varepsilon, \ \ t \geq 0.$$

Now, we will test the following estimate

$$||r(t)||_{\mathbf{H}^1} < c\varepsilon e^{-\delta t}, \ \ t \geq 0, \text{ where } 0 < \delta < \mu \text{ and } \mu = \text{Re } (\kappa - \sqrt{\kappa^2 - \eta}) > 0. \tag{5.5}$$

We will show by the sake of contradiction that the estimate (5.5) is satisfactory. Suppose that condition (5.5) is not satisfied at some time T_1, since $||r(0)||_{\dot{\mathbf{H}}^1} < \varepsilon$, we will have that by continuity of r in norm $\dot{\mathbf{H}}^1$ follows that

$$||r(t)||_{\dot{\mathbf{H}}^1} \leq c\varepsilon e^{-\delta t}, \ \ t \in [0, T_1], \tag{5.6}$$

then by (5.2) and (5.6) we will get the estimate:

$$\begin{aligned}
\|\mathcal{N}(v)(\tau)\|^2_{\dot{\mathbf{H}}^0} &= \sum_{|n|\geq 1} \left| \widehat{\mathcal{N}}(v)_n \right|^2 \\
&= \|\mathcal{N}(v)\mathcal{N}(h)\|^2_{\dot{\mathbf{H}}^0} \\
&\leq \|\mathcal{N}(v)\mathcal{N}(h)\|^2_{\mathbf{H}^0} \\
&\leq \|\mathcal{N}(v)\mathcal{N}(h)\|^2_{\mathbf{L}^2} \\
&\leq K\left(||v||^{2\rho}_{\mathbf{L}^\infty} + |h|^{2\rho}\right) ||r||^2_{\mathbf{L}^2} \\
&\leq K\varepsilon^{2+2\rho} e^{-2\delta t},
\end{aligned}$$

for all $t \in [0, T1]$. By hypothesis from g

$$||g(t,x)||_{\dot{\mathbf{H}}^0} \leq K\langle t\rangle^{-\frac{1}{\rho}-\frac{3}{2}}, \ \ t \geq 0,$$

we will have too

$$\begin{aligned}
||r(t)||_{\dot{\mathbf{H}}^1} &\leq K\varepsilon e^{-\mu t} + K\int_0^t e^{-\mu(t-\tau)}||\mathcal{N}(v)(t)||_{\dot{\mathbf{H}}^0} d\tau + K\int_0^t e^{-\mu(t-\tau)}||g(t,x)||_{\dot{\mathbf{H}}^0} d\tau \\
&\leq k\varepsilon e^{-\mu t} + K\varepsilon^{1+\rho} e^{-\mu t} \int_0^t e^{(-\mu-\delta)\tau} d\tau + K\int_0^t e^{-\mu(t-\tau)} \langle t\rangle^{-\frac{1}{\rho}-\frac{3}{2}} d\tau \\
&< K\varepsilon e^{-\delta t} + K\langle t\rangle^{-\frac{1}{\rho}-\frac{1}{2}}.
\end{aligned} \tag{5.7}$$

where $\varepsilon > 0$ is small enough. Then the estimate (4.6) is true for everything $t \geq 0$. We will consider the evolution of the average value of the solution

$$h''(t) + 2\kappa h'(t) = -\delta|h|^\rho h + \gamma(t),$$

where

$$\begin{aligned}
\gamma(t) &= |h|^\rho h - \frac{1}{2\pi}\int_\Omega |v(t,x)|^\rho v(t,x) dx + g_0(t,x) \\
|\gamma(t)| &\leq \left| \frac{1}{2\pi}\int_\Omega (|v(t,x)|^\rho v(t,x) - |h|^\rho h) dx \right| + ||g_0(t,x)||_{\dot{\mathbf{H}}^0} \\
&\leq K(||v||^\rho_{\mathbf{L}^2} + |h|^\rho)||r(t)||_{\mathbf{L}^\infty} + ||g_0(t,x)||_{\dot{\mathbf{H}}^0}
\end{aligned}$$

by the auxiliary lemma (2.4), page 362 of [2], we will have the estimate

$$\langle t\rangle^{\frac{1}{\rho}}|h(t)| + \langle t\rangle^{\frac{1}{\rho}+\frac{1}{2}}|h'(t)| \leq K.$$

This estimate with inequality (5.5) will gives us the estimate (5.1) for the solutions. □

6. **Asymptotic moderate solution**

Theorem 6.1. *We will assume that the initial data $\gamma \in \mathbf{H}^1$, $\chi \in \mathbf{L}^2$ are sufficiently small $||\gamma||_{\dot{\mathbf{H}}^1} + ||\chi||_{\mathbf{L}^2} \leq \varepsilon$. Moreover, the average value $\widehat{\gamma}_0 = 0$ and g satisf the inequality $||g||_{\mathbf{H}^s} \leq K\langle t\rangle^{-\frac{1}{\rho}-\frac{3}{2}}$ with $s > \frac{1}{2}$. So the next asymptotic relation is true.*

$$v(t,x) = At^{-\frac{1}{\rho}} + O(t^{-\frac{1}{\rho}-\frac{1}{2}}), \tag{6.1}$$

where $t \to \infty$ uniformly with respect to a $x \in \Omega$, where $A = 2\kappa(\frac{2\kappa}{\rho\delta})^{\frac{1}{\rho}}$.

Proof. As in the previous section by (5.4) and (5.7) and estimate of lemma 2.2, we will find that

$$\begin{aligned}||r(t)||_{\dot{\mathbf{H}}^1} &\leq K\varepsilon^{1+\frac{\rho}{2}}e^{-\mu t} + K\int_0^t e^{-\mu(t-\tau)}||\mathcal{N}(v)(\tau)||_{\dot{\mathbf{H}}^0}d\tau \\ &\leq K\varepsilon^{1+\frac{\rho}{2}}e^{-\mu t} + K\varepsilon^{1+\rho}e^{-\mu t}\int_0^t e^{(\mu-\delta)\tau}d\tau \\ &\leq K\varepsilon^{1+\frac{\rho}{2}}e^{-\delta t},\end{aligned}$$

Being small enough for everything $t \geq 0$. Then the estimate

$$||r(t)||_{\dot{\mathbf{H}}^1} \leq K\varepsilon^{1+\frac{\rho}{2}}e^{-\delta t},$$

is true for all $t \geq 0$. We will apply the second auxiliary statement (2.5) from [2], page 362, to get the estimates

$$\begin{aligned}|h(t)| &\leq K\varepsilon\langle t\rangle^{\frac{1}{\rho}} \quad \text{and} \\ |h'(t)| &\leq K\varepsilon^{1+\frac{\rho}{2}}\langle t\rangle^{-\frac{1}{\rho}-\frac{1}{2}},\end{aligned} \tag{6.2}$$

by replacing $z = h' + 2\kappa h$, we will rewrite equation (5.6) as

$$\frac{dz}{dt} = -\theta|z|^\rho z(1 - q(t)), \tag{6.3}$$

where $q(t) = \frac{\gamma(t)}{|z|^\rho z}$ and $\theta = \delta(2\kappa)^{-1-\rho}$

$$\begin{aligned}\gamma(t) &= |z|^\rho z - \frac{(2\kappa)^{1+\rho}}{2\pi}\int_\Omega |v(t,x)|^\rho v(t,x)dx - g_0(t,x), \\ z(0) &= h'(0) + 2\kappa h(0) \\ &= h'(0) + 2\kappa\varepsilon \\ &= 2\kappa\varepsilon - O(\varepsilon^{1+\frac{\rho}{2}}) > 0,\end{aligned}$$

and taking in account the estimate (6.2), we will have

$$\begin{aligned}|\gamma(t)| &= \left|\frac{(2\kappa)^{1+\rho}}{2\pi}\int_\Omega |v(t,x)|^\rho v(t,x)dx - |z|^\rho z - g_0(t,x)\right| \\ &\leq C\int_\Omega (|v(t,x)|^\rho + |h(t)^\rho)||r(t)||_{\dot{\mathbf{H}}^1} + K(|z|^\rho + |h(t)|^\rho)|h'(t)| + ||g_0(t,x)||_{\dot{\mathbf{H}}^1} \\ &\leq K\varepsilon^\rho\varepsilon^{1+\frac{1}{\rho}}e^{-\delta t} + K\varepsilon^{2\rho+1}\langle t\rangle^{-\frac{1}{\rho}-\frac{3}{2}} \\ &\leq K\varepsilon^{1+\frac{3}{2}\rho}\langle t\rangle^{-\frac{1}{\rho}-\frac{3}{2}},\end{aligned} \tag{6.4}$$

an now integrating with respect to time, we will have:

$$z(t) = z(0)(1 + \rho\theta|z(0)|^\rho(t - \int_0^t q(\tau)d\tau))^{-\frac{1}{\rho}}, \tag{6.5}$$

Furthermore we will estimate:

$$|q(t)| < \frac{1}{2}\langle T\rangle^{-\frac{1}{2}} \tag{6.6}$$

for all $t \geq 0$.

$$|q(0)| = \frac{|\gamma(t)|}{|z(0)|^{\rho} z(0)} = \frac{|\gamma(t)|}{|z(0)|^{\rho+1}} \leq \frac{K\varepsilon^{1+\frac{3}{2}\rho}}{\varepsilon^{1+\rho}} = K\varepsilon^{\frac{\rho}{2}},$$

By continuity in time, we will find a maximal time interval $[0, T]$, such that the estimate is true

$$|q(t)| \leq \frac{1}{2}\langle t\rangle^{-\frac{1}{2}},$$

for all $t \in [0, T]$. Then

$$|z(t)|^{1+\rho} \geq |z(0)|^{1+\rho}(1 + \frac{\rho\theta}{2}|z(0)|^{\rho} t)^{-1-\frac{1}{\rho}},$$

and by (6.4) we will have:

$$\begin{aligned} |q(t)| &\leq \frac{|\gamma(t)}{|z(0)|^{1+\rho}}(1 + \frac{\rho\theta}{2}|z(0)|^{\rho} t)^{-1-\frac{1}{\rho}} \\ &< K\varepsilon^{\frac{\rho}{2}}\langle t\rangle^{-\frac{1}{\rho}-\frac{3}{2}}(1 + \frac{\rho\theta}{2}\varepsilon^{\rho} t)^{\frac{1}{\rho}+1} \\ &< \frac{1}{2}\langle t\rangle^{-\frac{1}{2}}, \end{aligned}$$

for all $t \in [0, T]$. Since $\varepsilon > 0$ is small enough, therefore the estimate (6.4) is true all $t \geq 0$. By equations (6.5) and (6.6) the following is obtained asymptotic relationship

$$z(t) = (\rho\theta)^{-\frac{1}{\rho}} t^{-\frac{1}{\rho}} + O(t^{-\frac{1}{\rho}-\frac{1}{2}}).$$

By (6.4) and (6.5) we will have the following estimate:

$$\left| v(t, x) - \frac{z(t)}{2\kappa} \right| \leq |r(t, x)| + \left| \frac{h'(t)}{2\kappa} \right| \leq K\langle t\rangle^{-\frac{1}{\rho}-\frac{1}{2}},$$

then the asymptotic relation (6.1) is satisfied, this ends the test. □

7. Conclusion

In this article, an asymptotic solutions for a nonlinear damped wave problem was studied. We assured the order of convergence to zero of the analogous solution in the same way of the solution with g is identically zero. The problem of the extension of the solution for all positive time presented some difficulties that were solved in details here. We hope that this work can be useful for similar problems in this field. In future works we will study the inertial manifold associated with this equation and will focus on numerical implementations of this kind of equations using the asymptotic results obtained. Recently there is a lot of interest in studying these equations using Koopman operators, this would be one of our next objectives. Techniques of ergodic and functional analysis will be used.

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