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Desigualdades del tipo Minkowski y Hölder con una nueva integral fraccionariaa generalizada

Minkowski and Hölder type inequalities for a new generalized fractional integral.

Miguel Vivas-Cortez[a], Jorge Eliecer Hernández Hernández[b]

mjvivas@puce.edu.ec

jorgehernandez@ucla.edu.ve

[a]*Pontificia Universidad Católica del Ecuador, Facultad de Ciencias Naturales y Exactas, Escuela de Físicas y Matmáticas, Sede Quito, Ecuador*
[b]*Universidad Centroccidental Lisandro Alvarado, Decanato de Ciencias Administrativas y Empresariales, Departamento de Técnicas Cuantitativas, Barquisimeto, Venezuela*

Resumen

El presente estudio se refiere a algunas desigualdades de tipo Minkovski y Hölder usando un nuevo operador integral fraccional generalizado del tipo de Raina. Usando el modelo de función generalizada de Raina, $\mathcal{F}^{\sigma}_{\rho,\lambda}$, que involucran ciertos parámetros y una secuencia acotada de números reales positivos, se da una nueva definición de integral fraccional generalizada y de esto se deducen algunos otros operadores integrales fraccionarios clásicos. También la validez de los resultados principales en el marco de Riemman se comprueban las integrales fraccionarias de Liouville, Hadamard, Katugampola, Prabhakar y Salim.

Palabras claves: Desigualdades fraccionales integrales, Operador integral fraccionario generalizado
2015 MSC: 26A33, 26D15

Abstract

The present study is concerning about some inequalities of Minkoveski and Hölder type using a new generalized fractional integral operator of Raina's type. Using the Raina generalized function model, $\mathcal{F}^{\sigma}_{\rho,\lambda}$, which involve certain parameters and a bounded sequence of positive real numbers, a new definition of a generalized fractional integral is given and some others classical fractional integral operators are deduced from this. Also the validity of the main results in the setting of Riemman–Liouville, Hadamard, Katugampola, Prabhakar and Salim fractional integrals is proved.

Keywords: Fractional integral inequalities, generalized fractional integral operator
2015 MSC: 26A33, 26D15

1. Introduction

The study of integral inequalities has had a great evolution through the tools provided by fractional calculus during the last decades, and its application has been related to areas such as rheology, dynamic systems, biophysics, electrical networks, phenomena of blood flow, material science, science, mechanics, power, economy and control theory , see for more details [1, 3, 4, 6, 25, 27, 34]. In a work by Sonine (1869) dealing with arbitrary order differentiation we can find the first definition of the classical Riemman–Liouville fractional integral and later works led to the expression that until now is used

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{\alpha-1}}, \quad (x > a). \tag{1}$$

For easy application of fractional calculus mathematician have given many different definitions of fractional derivatives and integrals, out of which the most commonly used and invoked is the Riemann-Liouville operator. Some of the most studied and used in science are: Hadamard fractional integral [24], Katugampola fractional integral [23], Prabhakar fractional integral [28], Salim fractional integral [32]. During the twentieth century many authors have developed works leading to generalizations of this fractional integral, some of them add specific non-singular kernels, for example, Mittag-Leffler type functions. In particular, the works of: Prabhakar introduced in the kernel of the Riemann-Liouville fractional integral the three-parameters Mittag-Leffler function [28], Garra et al., in 2014, defined the Hilfer-Prabhakar fractional derivative, this fractional derivative is a generalization of Hilfer derivative in which Riemann-Liouville fractional integral is replaced by Prabhakar fractional integral [16], Atangana and Baleanu proposed the so-called AB fractional derivative operators which contain in the kernel the one-parameter Mittag-Leffler function [5], among others.

Following this model that modifies the nucleus of the Riemman Liouville integral, R. K. Raina introduces a type of fractional integral considering a more general nucleus in a study given some results determining similar properties for the generalized Wright's hypergeometric functions [31]. Also, there exist many integral inequalities related to the Raina fractional operators; the reader may refer to [10, 11, 20, 37] the aforementioned fractional integral operator was improved.

2. Preliminaries

Some definitions of fractional integrals have appeared later than the classic Riemman-Liouville fractional integral. Some of them are listed in the following.

The left and right Hadamard's fractional integral was introduced in [19] (1892)

$$\mathcal{H}_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{\alpha-1} \frac{f(t)dt}{t} \tag{2}$$

$$\mathcal{H}_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{t}{x}\right)^{\alpha-1} \frac{f(t)dt}{t} \tag{3}$$

for functions defined on an interval (a, b) with $0 \leq a < b < \infty$ and $\alpha > 0$.

In [23] the left and right Katugampola fractional integral are introduced as follows

$$ {}^{\varrho}I^{\alpha}_{a+}f(x) = \frac{\varrho^{\alpha-1}}{\Gamma(\alpha)}\int_a^x \frac{t^{\varrho-1}}{(x^{\varrho}-t^{\varrho})^{1-\alpha}}f(t)dt \quad (x>a), \tag{4}$$

$$ {}^{\varrho}I^{\alpha}_{b-}f(x) = \frac{\varrho^{\alpha-1}}{\Gamma(\alpha)}\int_x^b \frac{t^{\varrho-1}}{(x^{\varrho}-t^{\varrho})^{1-\alpha}}f(t)dt \quad (x<b). \tag{5}$$

for all complex-values Lebesgue measurable functions in $f \in X^p_c(a,b)$, $\alpha, \rho > 0$ and $0 \le a < b < \infty$, where $X^p_c(a,b)$, is the set of all complex-values Lebesgue measurable functions such that

$$\left(\int_a^b |t^c f(t)|^p \frac{dt}{t}\right)^{1/p} < \infty, \quad (c \in \mathbb{R}, 1 \le p < \infty)$$

or

$$ess \sup_{a\le t\le b} |tf(t)| < \infty, \quad (c \in \mathbb{R}, p = \infty).$$

When $\rho = 0$ we arrive to the Riemman–Liouville fractional integral and using L'Hopital rule we deduce that

$$\text{lím}_{\rho\to 0^+}({}^{\varrho}I^{\alpha}_{a+}f)(x) = \mathcal{H}^{\alpha}_{a+}f(x) \;\text{ and }\; \text{lím}_{\rho\to 0^+}({}^{\varrho}I^{\alpha}_{b-}f)(x) = \mathcal{H}^{\alpha}_{b-}f(x).$$

Many works have been published concerning this fractional integral with fractional integral inequalities [9, 14].

Also, in [21] the authors introduced a conformable fractional integral defined by means

$$\left({}^{\vartheta}_{\varpi}\mathcal{I}^{\mu}_{a+}\varphi\right)(x) = \int_a^x \left(\frac{x^{\vartheta+\varpi} - \xi^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\mu-1} \xi^{\vartheta+\varpi-1}\varphi(\xi)d\xi, \;\text{ for }\; (x>a)\,, \tag{6}$$

and

$$\left({}^{\vartheta}_{\varpi}\mathcal{I}^{\mu}_{a+}\varphi\right)(x) = \int_x^b \left(\frac{\xi^{\vartheta+\varpi} - x^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\mu-1} \xi^{\vartheta+\varpi-1}\varphi(\xi)d\xi, \;\text{ for }\; (x<b)\,, \tag{7}$$

where $\mu > 0$, $\vartheta \in (0,1], \varpi \in \mathbb{R} \ge 0$ such that $\vartheta + \varpi \neq 0$. For this conformable fractional integral the Chebyshev inequality and some properties of linearity, boundedness an some applications to differential equations are presented in the following works [21, 22].

Special functions have many relations with fractional calculus, and Mittag-Leffler function is a particularly significant one in this area [18]. For example, the left and right Prabhakar fractional integral is defined in [28] by means of

$$\epsilon_{a+}(\alpha,\beta,\eta,w)f(x) = \int_a^x (x-t)^{\beta-1}E^{\eta}_{\alpha,\beta}[w(x-t)^{\alpha}]f(t)dt \quad (x>a) \tag{8}$$

and

$$\epsilon_{b-}(\alpha,\beta,\eta,w)f(x) = \int_x^b (t-x)^{\beta-1}E^{\eta}_{\alpha,\beta}[w(t-x)^{\alpha}]f(t)dt \quad (x<b), \tag{9}$$

where $\alpha,\beta,\eta,w \in \mathbb{C}$, $Re(\alpha) > 0$, $Re(\beta) > 0$ and $E^{\eta}_{\alpha,\beta}$ is the Mittag-Leffler function

$$E^{\eta}_{\mu,\beta}(x) = \sum_{k=0}^{\infty} \frac{(\eta)_k}{\Gamma(\mu k+\beta)k!}x^k. \tag{10}$$

The left and right Salim fractional integral is introduced in [32] which is defined by means

$$I^{\gamma,\delta,q}_{\mu,\beta,w,p,a+}\wp_1(x) = \int_a^x (x-\xi)^{\beta-1} E^{\gamma,\delta,p}_{\mu,\beta,q}[w(x-\xi)^{\mu}]\wp_1(\xi)d\xi, \tag{11}$$

and

$$I^{\gamma,\delta,q}_{\mu,\beta,w,p,b-}\wp_1(x) = \int_x^b (\xi-x)^{\beta-1} E^{\gamma,\delta,p}_{\mu,\beta,q}[w(\xi-x)^{\mu}]\wp_1(\xi)d\xi, \tag{12}$$

where $\mu,\beta,\gamma,\delta,w \in \mathbb{C}$ and mín$\{\Re(\mu),\Re(\beta),\Re(\gamma),\Re(\delta)\} > 0$, $p,q > 0$, $q < \Re(\mu)+p$, where $E^{\gamma,\delta,p}_{\mu,\beta,q}$ is the modified Mittag-Leffler function

$$E^{\gamma,\delta,q}_{\mu,\beta,p}(x) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk}}{\Gamma(\mu k+\beta)(\delta)_{pk}} x^k. \tag{13}$$

Concerning the Prabhakar fractional integral several works have been published about properties, series representation, some applications to Lagenvin equations, among others [15, 16, 17, 29, 33]. Also, some estimations of fractional integral operators for convex functions and some fractional integral inequalities for generalized convex functions have been studied for the Salim fractional integral [8, 30, 32].

Meantime, the Raina fractional model is proposed in [2, 31] as one important models of fractional calculus and this is defined by integral similar to (1) but with a generalized Mittag-Leffler type function in the kernel. This is defined as follows.

Definition 2.1. *For any function φ which is L^1 on an interval $[a,b]$, the left and right Raina fractional integral operators applied to $\varphi(x)$ are defined by the following integral transforms, for $\lambda,\rho > 0$, $w \in R$:*

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi\right)(x) = \int_a^x (x-\xi)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x-\xi)^{\rho}\right]\varphi(\xi)d\xi, \quad (x>a), \tag{14}$$

and

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi\right)(x) = \int_x^b (\xi-x)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(\xi-x)^{\rho}\right]\varphi(\xi)d\xi, \quad (x<b), \tag{15}$$

where φ is such that the integral on the right side exits and $\mathcal{F}^{\sigma}_{\rho,\lambda}$ is the function, given by [31]:

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x) = \mathcal{F}^{\sigma(0),\sigma(1),\ldots}_{\rho,\lambda}(x) = \sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda)}x^k, \tag{16}$$

where $\rho,\lambda > 0, |x| < R$, and $\sigma = (\sigma(1),\ldots,\sigma(k),\ldots)$ is a bounded sequence in R, where R represents the set of real numbers.

Remark 2.2. *Observe that, if we select*

- *$\rho = 1, \lambda = 0$ and $\sigma(k) = ((\mu)_k(\beta)_k/(\gamma)_k)$ for $k = 0,1,2,...$ in* (16)*, then we get the classical Hypergeometric Function, that is*

  $$\mathcal{F}^{\sigma}_{\rho,\lambda}(x) = F(\mu,\beta;\gamma;x) = \sum_{k=0}^{\infty}\frac{(\mu)_k(\beta)_k}{(\gamma)_k k!}x^k,$$

  where μ,β and γ are parameters which can take arbitrary complex or real values such that $\gamma \neq 0,-1,-2,...$, and $(a)_k$ is the Pochhammer symbol given by

  $$(b_1)_k = \frac{\Gamma(b_1+k)}{\Gamma(b_1)} = b_1(b_1+1)\cdots(b_1+k-1), \quad k = 0,1,...,$$

  and restrict its domain to $|x| \leq 1$ (with $x \in \mathbb{C}$),

- *if we take* $\sigma(k) = (1, 1, 1, ...)$ *with* $\rho = \mu, (\Re(\mu) > 0)$, $\lambda = 1$ *and restricting its domain to* $x \in \mathbb{C}$ *in* (16)*, then we get the classical Mittag–Leffler function, defined as*

$$E_{\mu}(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\mu k + 1)} x^k.$$

- *if we take* $\sigma(k) = \frac{(\eta)_k}{k!}, \rho = \mu$ *and* $\lambda = \beta$*, we get the following extension of the Mittag-laffler function* (10) *defined in [28].*

- *if we take* $\sigma(k) = \frac{(\gamma)_{qk}}{(\delta)_{pk}}, \rho = \mu$ *and* $\lambda = \beta$ *in* (16)*, we get another extension of the Mittag–Leffler function* (13) *defined in [32].*

Remark 2.3. *It is easy to verify that* $\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi$ *and* $\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi$ *are bounded integral operators on* $L_p(a, b)$,$(1 \leq p \leq \infty)$*, if*

$$\mathfrak{M} := \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right] < \infty.$$

In fact, for $\varphi \in L_p(a, b)$ *we have*

$$\left\|\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi\right\|_p \leq \mathfrak{M}\left\|\varphi\right\|_p \quad \textit{and} \quad \left\|\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi\right\|_p \leq \mathfrak{M}\left\|\varphi\right\|_p,$$

where

$$\|\varphi\|_p = \left(\int_a^b |\varphi(x)|^p\, dx\right)^{1/p}.$$

Remark 2.4. *The importance of the Raina's operators stems indeed from their generality. That is, by specifying the coefficient* $\sigma(k)$*, we can obtain many useful fractional integral operators, as follows:*

- *if we choose* $\lambda = \mu, \sigma(0) = 1, \sigma(k) = 0$ *for* $k \neq 0$ *and* $w = 0$ *in Definition 2.1, we can deduce (left and right) Riemman–Liouville fractional integrals:*

$$\left(\mathcal{I}^{\mu}_{a+}\varphi\right)(x) = \frac{1}{\Gamma(\mu)}\int_a^x (x-\xi)^{\mu-1}\varphi(\xi)d\xi, \quad (x > a, \mu > 0); \tag{17}$$

$$\left(\mathcal{I}^{\mu}_{b-}\varphi\right)(x) = \frac{1}{\Gamma(\mu)}\int_x^b (\xi-x)^{\mu-1}\varphi(\xi)d\xi, \quad (x < b, \mu > 0). \tag{18}$$

- *if we choose* $\sigma(k) = \frac{(\eta)_k}{k!}, \rho = \alpha, \lambda = \beta$ *and* $a = 0$ *in Definition 2.1, we get the left and right Prabhakar fractional integrals* (8) *and* (9).

- *if we choose* $\sigma(k) = \frac{(\gamma)_{qk}}{(\delta)_{pk}}, \rho = \mu$ *and* $\lambda = \beta$ *in Definition 2.1, we get the left and right Salim fractional integrals* (11) *and* (12).

Th following Lemma will be useful for the development of this work and its prove is in [12].

Lemma 2.5. *Let h be a real concave function defined in the interval* $[a, b]$*. If* $x \in [a, b]$ *then*

$$h(a) + h(b) \leq h(b + a - x) + h(x) \leq 2h\left(\frac{a+b}{2}\right). \tag{19}$$

3. Main Results

Definition 3.1. *For any function $\varphi \in L_1([0,\infty])$, and any interval $[a,b] \subset [0,\infty]$ with $0 < a < b$, the conformable left and right Raina fractional integral operators applied to φ are defined by the following :*

$$\left({}_{\varpi}^{\vartheta}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi\right)(x) = \int_a^x \left(\frac{x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\lambda-1} \xi^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi})^{\rho}\right]\varphi(\xi)d\xi, \quad \text{for} \quad (x>a), \tag{20}$$

and

$$\left({}_{\varpi}^{\vartheta}\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi\right)(x) = \int_x^b \left(\frac{\xi^{\vartheta+\varpi}-x^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\lambda-1} \xi^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(\xi^{\vartheta+\varpi}-x^{\vartheta+\varpi})^{\rho}\right]\varphi(\xi)d\xi, \quad (x<b), \tag{21}$$

where $\lambda,\rho > 0$, $w \in \mathbb{R}$, $\vartheta \in (0,1], \varpi \in \mathbb{R} \geq 0$, $\vartheta+\varpi \neq 0$, σ a bounded arbitrary sequence of real (or complex) numbers and φ is such that the integral on the right side exits, with $\mathcal{F}^{\sigma}_{\rho,\lambda}$ as the function given in [31].

Remark 3.2. *From this definition we obtain some of the following fractional integral:*

1. *If $\vartheta = 1$ and $\varpi = 0$ then we obtain the left and right Raina's fractional integral given in* (14) *and* (15)
2. *if we choose $\vartheta = 1$, $\varpi = 0$ and $\lambda = \mu, \sigma(0) = 1, \sigma(k) = 0$ for $k \neq 0$ and $w = 0$, we can deduce the Riemann–Liouville fractional integrals (left and right)* (17) *and* (18).
3. *if we choose $\vartheta = 1$, $\varpi = 0$, and $\sigma(k) = \frac{(\eta)_k}{k!}, \rho = \mu, \lambda = \beta$ and $a = 0$ we get the Prabhakar fractional integral given by* (8) *and* (9).
4. *if we choose $\vartheta = 1$, $\varpi = 0$ and $\sigma(k) = \frac{(\gamma)_{qk}}{(\delta)_{pk}}, \rho = \mu$ and $\lambda = \beta$, we get the well-known Salim fractional integral given by* (11) *and* (12).
5. *if we choose $\lambda = \mu, \sigma(0) = 1, \sigma(k) = 0$ for $k \neq 0$ and $w = 0$, we can deduce the left and right Khan fractional integrals given by* (6) *and* (7).
6. *If we choose $\varrho = \vartheta + \varpi > 0$ and $\lambda = \mu, \sigma(0) = 1, \sigma(k) = 0$ for $k \neq 0$ and $w = 0$, we can deduce the left and right Katugampola fractional integrals given by* (4) *and* (5).

Theorem 3.3. *Let $\lambda,\rho > 0, p \geq 1$, $w \in \mathbb{R}$, $\vartheta \in (0,1], \varpi \in \mathbb{R} > 0$, $\vartheta + \varpi \neq 0$ and $\sigma = \{\sigma_k\}_{k=1}^{\infty}$ a bounded arbitrary sequence of real (or complex) numbers. Let f, g be real functions defined on $[a,\infty)$, such that $\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f^p(x) < \infty$ and $\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}g^p(x) < \infty$. If $0 < m \leq (f(t)/g(t)) \leq M, t \in [a,x]$, then the following inequality holds*

$$\left[\left({}_{\varpi}^{\vartheta}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f^p\right)(x)\right]^{1/p} + \left[\left({}_{\varpi}^{\vartheta}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}g^p\right)(x)\right]^{1/p} \leq \left(\frac{1+M(m+2)}{(M+1)(m+1)}\right)\left[\left({}_{\varpi}^{\vartheta}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}(f+g)^p\right)(x)\right]^{1/p}. \tag{22}$$

Proof. Since $f(t)/g(t) < M, t \in [a,x]$, $x > 0$ then we have

$$f(t) \leq Mg(t) + Mf(t) - Mf(t)$$

and so

$$(M+1)^p f^p(t) \leq M^p (f+g)^p (t).$$

Multiplying both side by $\left(\frac{x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\lambda-1}\xi^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi})^{\rho}\right]$ we obtain

$$\begin{aligned}\left(\frac{x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\lambda-1}&\xi^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi})^{\rho}\right](M+1)^{p}f^{p}(t)\\&\leq\left(\frac{x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\lambda-1}\xi^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi})^{\rho}\right]M^{p}\left(f+g\right)^{p}(t).\end{aligned}$$

Integrating over $t\in[a,x]$ we have

$$\begin{aligned}(M+1)^{p}\int_{a}^{x}&\left(\frac{x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\lambda-1}\xi^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi})^{\rho}\right]f^{p}(t)\\&\leq M^{p}\int_{a}^{x}\left(\frac{x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\lambda-1}\xi^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi})^{\rho}\right](f+g)^{p}(t).\end{aligned}$$

which is equivalet to

$$\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f^{p}\right)(x)\leq\frac{M^{p}}{(M+1)^{p}}\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\left(f+g\right)^{p}\right)(x),$$

therefore

$$\left[\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f^{p}\right)(x)\right]^{1/p}\leq\frac{M}{(M+1)}\left[\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\left(f+g\right)^{p}\right)(x)\right]^{1/p}.\tag{23}$$

Now, using the fact that $mg(t)\leq f(t)$, we obtain

$$\left(1+\frac{1}{m}\right)g(t)\leq\frac{1}{m}\left(f+g\right)(t),$$

similarly we have

$$\left[\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}g^{p}\right)(x)\right]^{1/p}\leq\frac{1}{(m+1)}\left[\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\left(f+g\right)^{p}\right)(x)\right]^{1/p}.\tag{24}$$

Adding (23) and (24) the desired inequality (22) is obtained. ■

Remark 3.4. *Letting $\lambda=\alpha,\sigma=(1,0,0,0,...),w=0$ y $a=0$ en el Theorem 3.3 we have*

$$\left[\left(I^{\alpha}_{0+}f^{p}\right)(x)\right]^{1/p}+\left[\left(I^{\alpha}_{0+}g^{p}\right)(x)\right]^{1/p}\leq\left(\frac{1+M(m+2)}{(M+1)(m+1)}\right)\left[\left(I^{\alpha}_{0+}\left(f+g\right)^{p}\right)(x)\right]^{1/p}$$

wich coincides with Theorem 2.1 en [12]. In particular, if $\alpha=1$ then

$$\|f\|_{p}+\|g\|_{p}\leq\left(\frac{1+M(m+2)}{(M+1)(m+1)}\right)\|f+g\|_{p},$$

making coincidence with Theorem 1.2 in [7]. With the choice of parameters given in numeral 3 of Remark 3.2 in Theorem 1, we obtain the inequality for the left Prabhakar fractional integral

$$\begin{aligned}\left[\left(\epsilon_{a+}(\alpha,\beta,\eta,w)f^{p}\right)(x)\right]^{1/p}&+\left[\left(\epsilon_{a+}(\alpha,\beta,\eta,w)g^{p}\right)(x)\right]^{1/p}\\&\leq\left(\frac{1+M(m+2)}{(M+1)(m+1)}\right)\left[\left(\epsilon_{a+}(\alpha,\beta,\eta,w)(f+g)^{p}\right)(x)\right]^{1/p}.\end{aligned}$$

Also, using the appropriate choices of the parameters given in numeral 4 (Remark 3.2) in Theorem 1, we have the following inequality for the left Salim fractional integral

$$\left[\left(I^{\gamma,\delta,q}_{\mu,\beta,w,p,a+}f^p\right)(x)\right]^{1/p}+\left[\left(I^{\gamma,\delta,q}_{\mu,\beta,w,p,a+}g^p\right)(x)\right]^{1/p}\leq\left(\frac{1+M(m+2)}{(M+1)(m+1)}\right)\left[\left(I^{\gamma,\delta,q}_{\mu,\beta,w,p,a+}(f+g)^p\right)(x)\right]^{1/p}.$$

Additionally, using the choices of the parameters indicated in numeral 5 (Remark 3.2) in Theorem 1 we obtain the inequality for the left Katugampola fractional integral

$$\left[\left({}^{\varrho}I^{\alpha}_{a+}f^p\right)(x)\right]^{1/p}+\left[\left({}^{\varrho}I^{\alpha}_{a+}g^p\right)(x)\right]^{1/p}\leq\left(\frac{1+M(m+2)}{(M+1)(m+1)}\right)\left[\left({}^{\varrho}I^{\alpha}_{a+}(f+g)^p\right)(x)\right]^{1/p},$$

taking limit when $\varrho\to 0^+$ we obtain the inequality for the Hadamard fractional integral

$$\left[\left(H^{\alpha}_{1}f^p\right)(x)\right]^{1/p}+\left[\left(H^{\alpha}_{1}g^p\right)(x)\right]^{1/p}\leq\left(\frac{1+M(m+2)}{(M+1)(m+1)}\right)\left[\left(H^{\alpha}_{1}(f+g)^p\right)(x)\right]^{1/p}$$

wich coincides with Theorem 6 in [35]. Also, for the Khan fractional integral we have

$$\left[\left({}^{\vartheta}_{\varpi}I^{\mu}_{a+}f^p\right)(x)\right]^{1/p}+\left[\left({}^{\vartheta}_{\varpi}I^{\mu}_{a+}g^p\right)(x)\right]^{1/p}\leq\left(\frac{1+M(m+2)}{(M+1)(m+1)}\right)\left[\left({}^{\vartheta}_{\varpi}I^{\mu}_{a+}(f+g)^p\right)(x)\right]^{1/p},$$

Theorem 3.5. *Let $p\geq 1,\lambda,\rho>0$, $w\in R$, $\vartheta\in(0,1],\varpi\in\mathbb{R}>0$, $\vartheta+\varpi\neq 0$ and $\sigma=(\sigma_k)_{k=1}^{\infty}$ a bounded arbitrary sequence of real (or complex) numbers.Let f,g be real functions defined on $[a,\infty)$, such that $\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f^p\right)(x)<\infty$ y $\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}g^p\right)(x)<\infty$. If $0<m\leq(f(t)/g(t))\leq M, t\in[a,x]$, then the following inequality holds*

$$\left[\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f^p\right)(x)\right]^{2/p}+\left[\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}g^p\right)(x)\right]^{2/p}\geq\left(\frac{(M+1)(m+1)}{M}-2\right)\left[\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f^p\right)(x)\right]^{1/p}\left[\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}g^p\right)(x)\right]^{1/p}. \tag{25}$$

Proof. Multiplying (23) and (24) it follows that

$$\frac{(M+1)(m+1)}{M}\left[\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f^p\right)(x)\right]^{1/p}\left[\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}g^p\right)(x)\right]^{1/p}\leq\left[\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}(f+g)^p\right)(x)\right]^{2/p}. \tag{26}$$

By an application of the Minkowski inequality in the right side of (26), we have

$$\left[\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}(f+g)^p\right)(x)\right]^{2/p}\leq\left(\left[\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f^p\right)(x)\right]^{1/p}+\left[\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}g^p\right)(x)\right]^{1/p}\right)^2.$$

It follows that

$$\left[\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}(f+g)^{p}\right)(x)\right]^{2/p} \tag{27}$$

$$\leq \left[\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f^{p}\right)(x)\right]^{2/p} + \left[\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}g^{p}\right)(x)\right]^{2/p}$$

$$+2\left[\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f^{p}\right)(x)\right]^{1/p}\left[\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}g^{p}\right)(x)\right].^{1/p}$$

Using the inequalities (26) and (27) the desired result (25) is obtained . ■

Remark 3.6. *Letting $\lambda=\alpha, \sigma=(1,0,0,...), w=0$ y $a=0$ in Theorem 3.5 we have*

$$\left[\left(I^{\alpha}_{0+}f^{p}\right)(x)\right]^{2/p} + \left[\left(I^{\alpha}_{0+}g^{p}\right)(x)\right]^{2/p} \geq \left(\frac{(M+1)(m+1)}{M}-2\right)\left[\left(I^{\alpha}_{0+}f^{p}\right)(x)\right]^{1/p}\left[\left(I^{\alpha}_{0+}g^{p}\right)(x)\right]^{1/p}$$

which coincides with Theorem 2.3 in [12]. In particulr, if $\alpha=1$ then

$$\|f\|_p^2+\|g\|_p^2 \geq \left(\frac{(M+1)(m+1)}{M}-2\right)\|f\|_p\,\|g\|_p$$

making coincidence with Theorem 2.2 in [13]. Following Remark 3.4 we have the result in Theorem 3.5 for Prabhakar and Salim fractional integrals

$$[(\epsilon_{a+}(\alpha,\beta,\eta,w)f^{p})(x)]^{2/p} + [(\epsilon_{a+}(\alpha,\beta,\eta,w)g^{p})(x)]^{2/p}$$

$$\geq \left(\frac{(M+1)(m+1)}{M}-2\right)[(\epsilon_{a+}(\alpha,\beta,\eta,w)f^{p})(x)]^{1/p}\,[(\epsilon_{a+}(\alpha,\beta,\eta,w)g^{p})(x)]^{1/p}.$$

and

$$\left[\left(I^{\gamma,\delta,q}_{\mu,\beta,w,p,a+}f^{p}\right)(x)\right]^{2/p} + \left[\left(I^{\gamma,\delta,q}_{\mu,\beta,w,p,a+}g^{p}\right)(x)\right]^{2/p}$$

$$\geq \left(\frac{(M+1)(m+1)}{M}-2\right)\left[\left(I^{\gamma,\delta,q}_{\mu,\beta,w,p,a+}f^{p}\right)(x)\right]^{1/p}\left[\left(I^{\gamma,\delta,q}_{\mu,\beta,w,p,a+}g^{p}\right)(x)\right]^{1/p}.$$

respectively. Also for the Katugampola fractional integral we have

$$[({}^{\varrho}I^{\alpha}_{a+}f^{p})(x)]^{2/p} + [({}^{\varrho}I^{\alpha}_{a+}g^{p})(x)]^{2/p}$$

$$\geq \left(\frac{(M+1)(m+1)}{M}-2\right)[({}^{\varrho}I^{\alpha}_{a+}f^{p})(x)]^{1/p}\,[({}^{\varrho}I^{\alpha}_{a+}g^{p})(x)]^{1/p},$$

and letting $\varrho\to 1$ we have the result for the Hadamard fractional integral

$$\left[\left(H^{\alpha}_{1}f^{p}\right)(x)\right]^{2/p} + \left[\left(H^{\alpha}_{1}g^{p}\right)(x)\right]^{2/p}$$

$$\geq \left(\frac{(M+1)(m+1)}{M} - 2\right) \left[\left(H_1^{\alpha} f^p\right)(x)\right]^{1/p} \left[\left(H_1^{\alpha} g^p\right)(x)\right]^{1/p}.$$

Similarly for the Khan fractional integral we have

$$\left[\left({}_{varpi}^{\vartheta} I_{a+}^{\mu} f^p\right)(x)\right]^{2/p} + \left[\left({}_{\varpi}^{\vartheta} I_{a+}^{\mu} g^p\right)(x)\right]^{2/p}$$

$$\geq \left(\frac{(M+1)(m+1)}{M} - 2\right) \left[\left({}_{\varpi}^{\vartheta} I_{a+}^{\mu} f^p\right)(x)\right]^{1/p} \left[\left({}_{\varpi}^{\vartheta} I_{a+}^{\mu} g^p\right)(x)\right]^{1/p},$$

Theorem 3.7. *Let $p, q \in \mathbb{R}$ with $p > 1$ and $1/p + 1/q = 1$. Let $\lambda, \rho > 0$, $w \in \mathbb{R}$, $\vartheta \in (0,1], \varpi \in \mathbb{R} > 0$, $\vartheta + \varpi \neq 0$ and $\sigma = (\sigma_k)_{k=1}^{\infty}$ a bounded arbitrary sequence of real numbers. If $f, g : a, b \to \mathbb{R}$ are functions such that $f^p, g^q, fg \in L_1([a,b])$ and*

$$0 < m \leq \frac{f(x)}{g(x)} \leq M, \textit{ for any } x \in [a,b], a, b \in [0,\infty)$$

for all $t \in [a,x]$, then

$$\left(\left({}_{\varpi}^{\vartheta}\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma} f\right)(x)\right)^{1/p} \left(\left({}_{\varpi}^{\vartheta}\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma} g\right)(x)\right)^{1/q} \leq \left(\frac{M}{m}\right)^{1/pq} \left({}_{\varpi}^{\vartheta}\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma} f^{1/p} g^{1/q}\right)(x).$$

Proof. Using the fact that $\frac{f(x)}{g(x)} \leq M$ for all $t \in [a,x]$, we have

$$g^{1/q}(t) \geq M^{-1/q} f^{1/q}(t),$$

so, multiplying in both sides by $f^{1/p}$ we get

$$f^{1/p}(t) g^{1/q}(t) \geq M^{-1/q} f(t). \tag{28}$$

Multiplying both sides of (28) by $\left(\frac{x^{\vartheta+\varpi} - \xi^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\lambda-1} \xi^{\vartheta+\varpi-1} \mathcal{F}_{\rho,\lambda}^{\sigma}\left[w(x^{\vartheta+\varpi} - \xi^{\vartheta+\varpi})^{\rho}\right]$ and integrating over $t \in [a,x]$we obtain

$$\int_a^x \left(\frac{x^{\vartheta+\varpi} - \xi^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\lambda-1} \xi^{\vartheta+\varpi-1} \mathcal{F}_{\rho,\lambda}^{\sigma}\left[w(x^{\vartheta+\varpi} - \xi^{\vartheta+\varpi})^{\rho}\right] f^{1/p}(t) g^{1/q}(t) dt$$
$$\geq M^{-1/q} \int_a^x \left(\frac{x^{\vartheta+\varpi} - \xi^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\lambda-1} \xi^{\vartheta+\varpi-1} \mathcal{F}_{\rho,\lambda}^{\sigma}\left[w(x^{\vartheta+\varpi} - \xi^{\vartheta+\varpi})^{\rho}\right] f(t) dt,$$

from which the following is obtained

$$\left(\left({}_{\varpi}^{\vartheta}\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma} f^{1/p} g^{1/q}\right)(x)\right)^{1/p} \geq M^{-1/pq} \left(\left({}_{\varpi}^{\vartheta}\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma} f\right)(x)\right)^{1/p}. \tag{29}$$

Similarly, using the lower bound $m \leq \frac{f(x)}{g(x)}$ we have

$$f^{1/p}(t) g^{1/q}(t) \geq m^{1/p} g(t). \tag{30}$$

So, Multiplying both sides of (30) by $\left(\frac{x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\lambda-1}\xi^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi})^{\rho}\right]$ and integrating over $t\in[a,x]$we obtain

$$\left(\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f^{1/p}g^{1/q}\right)(x)\right)^{1/q}\geq m^{1/pq}\left(\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}g\right)(x)\right)^{1/q}. \tag{31}$$

Therefore from inequalities (29) and (31) we obtain the desired result

$$\left(\frac{M}{m}\right)^{1/pq}\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f^{1/p}g^{1/q}\right)(x)\geq\left(\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f\right)(x)\right)^{1/p}\left(\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}g\right)(x)\right)^{1/q}. \tag{32}$$

The proof is complete. ■

Remark 3.8. *Letting $\lambda=\alpha,\sigma=(1,0,0,...),w=0$ in Theorem 3.7 we have*

$$\left(\left(I^{\alpha}_{a+}f\right)(x)\right)^{1/p}\left(\left(I^{\alpha}_{a+}g\right)(x)\right)^{1/q}\leq\left(\frac{M}{m}\right)^{1/pq}\left(I^{\alpha}_{a+}f^{1/p}g^{1/q}\right)(x)$$

which coincides with Theorem 2.3 in [12]. In particular, if $\alpha=1$ then

$$\left(\int_a^x f(t)dt\right)^{1/p}\left(\int_a^x f(t)dt\right)^{1/q}\leq\left(\frac{M}{m}\right)^{1/pq}\int_a^x f^{1/p}(t)g^{1/q}(t)dt.$$

Following Remarks 3.4 and 3.6 we obtain the result in Theorem 3.7 for Prabhakar, Salim, Katugampola, Hadamard and Khan fractional integrals.

Theorem 3.9. *Let $p,q\in\mathbb{R}$ with $p>1$ and $1/p+1/q=1$. Let $\lambda,\rho>0$, $w\in\mathbb{R}$, $\vartheta\in(0,1],\varpi\in\mathbb{R}>0$, $\vartheta+\varpi\neq0$ and $\sigma=(\sigma_k)_{k=1}^{\infty}$ a bounded arbitrary sequence of real numbers. Let $f,g:a,b\to\mathbb{R}$ be two positive functions such that such that $\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f^{p}\right)(x)<\infty$ y $\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}g^{p}\right)(x)<\infty$. If*

$$0<c<m\leq\frac{rf(t)}{g(t)}\leq M,\ \textit{for any}\ t\in[a,x], \tag{33}$$

then

$$\begin{aligned}\frac{M+r}{r(M-c)}&\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\left(rf(t)-cg(t)\right)^{p}(x)\right)^{1/p}\\&\leq\left(\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}g^{p}\right)(x)\right)^{1/p}+\left(\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f^{p}\right)(x)\right)^{1/p}\\&\leq\frac{m+r}{r(m-c)}\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\left(rf(t)-cg(t)\right)^{p}(x)\right)^{1/p}\end{aligned} \tag{34}$$

Proof. From the given condition (33) we get

$$m-c<\frac{rf(t)}{g(t)}-c<M-c,$$

and from here we obtain

$$\frac{1}{(M-c)^p}\left(rf(t)-cg(t)\right)^p\leq g^p(t)\leq\frac{1}{(m-c)^p}\left(rf(t)-cg(t)\right)^p.$$

Multiplying by $\left(\frac{x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\lambda-1}\xi^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi})^{\rho}\right]$ and integrating over $t\in[a,x]$we obtain

$$\begin{aligned}\frac{1}{M-c}\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\left(rf(t)-cg(t)\right)^{p}(x)\right)^{1/p} &\leq \left(\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}g^{p}\right)(x)\right)^{1/p} \\ &\leq \frac{1}{m-c}\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\left(rf(t)-cg(t)\right)^{p}(x)\right)^{1/p}\end{aligned} \tag{35}$$

Again from condition (33) we obtain

$$\frac{-1}{m}\leq-\frac{g(t)}{rf(t)}\leq\frac{-1}{M}$$

then

$$\frac{1}{c}-\frac{1}{m}\leq\frac{1}{c}-\frac{g(t)}{rf(t)}\leq\frac{1}{c}-\frac{1}{M}$$

which gives

$$\frac{m-c}{mc}\leq\frac{rf(t)-cg(t)}{crf(t)}\leq\frac{M-c}{Mc}.$$

From here it follows that

$$\left(\frac{M}{r(M-c)}\right)^{p}\left(rf(t)-cg(t)\right)^{p}\leq g^{p}(t)\leq\left(\frac{m}{r(m-c)}\right)^{p}\left(rf(t)-cg(t)\right)^{p}.$$

Multiplying by $\left(\frac{x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\lambda-1}\xi^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi})^{\rho}\right]$ and integrating over $t\in[a,x]$we obtain

$$\begin{aligned}\frac{M}{r(M-c)}\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\left(rf(t)-cg(t)\right)^{p}(x)\right)^{1/p} &\leq \left(\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f^{p}\right)(x)\right)^{1/p} \\ &\leq \frac{m}{r(m-c)}\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\left(rf(t)-cg(t)\right)^{p}(x)\right)^{1/p}\end{aligned} \tag{36}$$

Adding (35) and (36) we obtain the desired result (34). ■

Remark 3.10. *The inequality in Theorem 3.9 remains valid for all the fractional integrals mentioned in Remark 3.4.*

In the proof of the following theorem we will use the classical Young inequality:

$$xy\leq\frac{x^{p}}{p}+\frac{y^{q}}{q},\ \text{ para cualesquiera } x,y\geq0 \text{ y } p>1,\frac{1}{p}+\frac{1}{q}=1.$$

Theorem 3.11. *Let $p,q\in\mathbb{R}$ with $p>1$ and $1/p+1/q=1$. Let $\lambda,\rho>0$, $w\in\mathbb{R}$, $\vartheta\in(0,1]$, $\varpi\in\mathbb{R}>0$, $\vartheta+\varpi\neq0$ and $\sigma=(\sigma_k)_{k=1}^{\infty}$ a bounded arbitrary sequence of real numbers. If $f,g:a,b\to\mathbb{R}$ are functions such that $f^{p},g^{q},fg\in L_1([a,b])$ and*

$$0<m\leq\frac{f(x)}{g(x)}\leq M,\ \textit{for any } x\in[a,b]\,,a,b\in[0,\infty)$$

then

$$\begin{aligned}\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}fg\right)(x) &\leq \frac{2^{p-1}}{p}\left(\frac{M}{M+1}\right)^{p}\left(\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f^{p}\right)(x)+\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}g^{p}\right)(x)\right) \\ &\quad+\frac{2^{q-1}}{q}\left(\frac{1}{m+1}\right)^{q}\left(\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f^{q}\right)(x)+\left({}^{\vartheta}_{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}g^{q}\right)(x)\right).\end{aligned}$$

Proof. Since $0 < m \leq f(x)/g(x) \leq M$, for any $x \in [a, b]$, we have

$$\begin{aligned} f(x) &\leq \frac{M}{M+1}\left(f(x)+g(x)\right), \\ g(x) &\leq \frac{1}{m+1}\left(f(x)+g(x)\right). \end{aligned}$$

Using the Young inequality

$$f(x)g(x) \leq \frac{f(x)^p}{p} + \frac{g(x)^q}{q} \leq \frac{1}{p}\left(\frac{M}{M+1}\right)^p (f(x)+g(x))^p + \frac{1}{q}\left(\frac{1}{m+1}\right)^q (f(x)+g(x))^q$$

then, multiplying by $\left(\frac{x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\lambda-1} \xi^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi})^{\rho}\right]$ we obtain

$$\begin{aligned} &\int_a^x \left(\frac{x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\lambda-1} \xi^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi})^{\rho}\right] f(t)g(t)dt \\ &\leq \frac{1}{p}\left(\frac{M}{M+1}\right)^p \int_a^x \left(\frac{x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\lambda-1} \xi^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi})^{\rho}\right] (f(t)+g(t)^p\, dt \\ &+ \frac{1}{q}\left(\frac{1}{m+1}\right)^q \int_a^x \left(\frac{x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\lambda-1} \xi^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi})^{\rho}\right] (f(t)+g(t))^q dt \end{aligned}$$

Using the classical inequality: $(c+d)^p \leq 2^{p-1}\left(c^p+d^p\right)$, we obtain

$$\begin{aligned} &\int_0^x \left(\frac{x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\lambda-1} \xi^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi})^{\rho}\right] f(t)g(t)dt \\ &\leq \frac{1}{p}\left(\frac{M}{M+1}\right)^p 2^{p-1}\times \\ &\qquad \int_a^x \left(\frac{x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\lambda-1} \xi^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi})^{\rho}\right] (f^p(t)+g^p(t))dt \\ &+ \frac{1}{q}\left(\frac{1}{m+1}\right)^q 2^{q-1}\times \\ &\qquad \int_a^x \left(\frac{x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\lambda-1} \xi^{\vartheta+\varpi-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x^{\vartheta+\varpi}-\xi^{\vartheta+\varpi})^{\rho}\right] (f^q(t)+g^q(t))\, dt \\ &= \frac{1}{p}\left(\frac{M}{M+1}\right)^p 2^{p-1}\left(\tfrac{\vartheta}{\varpi}\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f^p\right)(x) + \left(\tfrac{\vartheta}{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}g^p\right)(x)\right) \\ &\qquad + \frac{1}{q}\left(\frac{1}{m+1}\right)^q 2^{q-1}\left(\left(\tfrac{\vartheta}{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f^q\right)(x) + \left(\tfrac{\vartheta}{\varpi}\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}g^q\right)(x)\right). \end{aligned}$$

The proof is complete. ■

Remark 3.12. *Letting $\lambda = \alpha, w = 0$ y $\sigma = (1, 0, 0, ...)$ in Theorem 3.11 the following fractional integrals of Riemann-Liouville type is obtained*

$$\begin{aligned} \left(I^{\alpha}_{a+}fg\right)(x) \leq{}& \frac{2^{p-1}}{p}\left(\frac{M}{M+1}\right)^p \left(\left(I^{\alpha}_{0+}f^p\right)(x) + \left(I^{\alpha}_{0+}g^p\right)(x)\right) \\ &+ \frac{2^{q-1}}{q}\left(\frac{1}{m+1}\right)^q \left(\left(I^{\alpha}_{0+}f^q\right)(x) + \left(I^{\alpha}_{0+}g^q\right)(x)\right). \end{aligned}$$

In particular, if $\alpha = 1$ se have

$$\int_0^x f(t)g(t)dt \leq \frac{2^{p-1}}{p}\left(\frac{M}{M+1}\right)\left(\|f\|_p^p + \|g\|_p^p\right) + \frac{2^{q-1}}{q}\left(\frac{1}{m+1}\right)^q\left(\|f\|_q^q + \|g\|_q^q\right).$$

making coincidence with Teorema 2.4 in [13]. With a suitable choices of the parameters in Definition 3.1 the inequality in Theorem 3.11 remains valid for the classical Riemman integral, Riemman–Liouville, Hadamard, Prabhakar, Salim and Khan fractional integral.

Theorem 3.13. *Let $p, q \in \mathbb{R}$ with $p > 1$. Let $\lambda, \rho > 0$, $w \in \mathbb{R}$, $\vartheta \in (0,1], \varpi \in \mathbb{R} > 0$, $\vartheta + \varpi \neq 0$ and $\sigma = (\sigma_k)_{k=1}^{\infty}$ a bounded arbitrary sequence of real numbers. Let $f, g : a, b \to \mathbb{R}$ be two positive functions such that such that $\left({}_{\varpi}^{\vartheta}\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma} f^p\right)(x) < \infty$ y $\left({}_{\varpi}^{\vartheta}\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma} g^p\right)(x) < \infty$. If*

$$0 < m_1 \leq f(t) \leq M_1, \text{ and } 0 < m_2 \leq g(t) \leq M_2$$

for $t \in [a, x]$, then the following inequality holds

$$\begin{aligned}\left[\left({}_{\varpi}^{\vartheta}\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma} f^p\right)(x)\right]^{1/p} &+ \left[\left({}_{\varpi}^{\vartheta}\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma} g^p\right)(x)\right]^{1/p} \\ &\leq M\left[\left({}_{\varpi}^{\vartheta}\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma} (f+g)^p\right)(x)\right]^{1/p}.\end{aligned} \tag{37}$$

where

$$M = \frac{M_1(m_1 + M_2) + M_2(m_2 + M_1)}{(m_1 + M_2)(m_2 + M_1)}$$

Proof. From the given condition for g we have that

$$\frac{1}{M_2} \leq \frac{1}{g(t)} \leq \frac{1}{m_2}$$

and using the condition for $f(t)$ we obtain

$$\frac{m_1}{M_2} \leq \frac{f(t)}{g(t)} \leq \frac{M_1}{m_2}.$$

Easily we obtain

$$g(t)\left(1 + \frac{M_2}{m_1}\right) \leq \frac{M_2}{m_1}(f(t) + g(t)).$$

It follows that

$$g^p(t) \leq \left(\frac{M_2}{m_1 + M_2}\right)^p (f(t) + g(t))^p. \tag{38}$$

From the condition $\frac{f(t)}{g(t)} \leq \frac{M_1}{m_2}$ we obtain

$$f^p(t) \leq \left(\frac{M_1}{m_1 + M_2}\right)^p (f(t) + g(t))^p. \tag{39}$$

Multiplying (38) and (39) by $\left(\frac{x^{\vartheta+\varpi} - \xi^{\vartheta+\varpi}}{\vartheta+\varpi}\right)^{\lambda-1} \xi^{\vartheta+\varpi-1}\mathcal{F}_{\rho,\lambda}^{\sigma}\left[w(x^{\vartheta+\varpi} - \xi^{\vartheta+\varpi})^{\rho}\right]$, integrating over $t \in [a, x]$ and adding the resulting inequalities we obtain

$$\begin{aligned}\left[\left({}_{\varpi}^{\vartheta}\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma} f^p\right)(x)\right]^{1/p} &+ \left[\left({}_{\varpi}^{\vartheta}\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma} g^p\right)(x)\right]^{1/p} \\ &\leq M\left[\left({}_{\varpi}^{\vartheta}\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma} (f+g)^p\right)(x)\right]^{1/p}.\end{aligned}$$

where

$$M = \frac{M_1(m_1 + M_2) + M_2(m_2 + M_1)}{(m_1 + M_2)(m_2 + M_1)}.$$

The proof is complete. ■

Remark 3.14. *The inequality in Theorem 3.13 remains valid for all the fractional integrals mentioned in Remark 3.4.*

Conclusion

In this article, some fractional integral inequalities of Minkowski and Hölder type (Theorems 1,2,3,4,5 and 6) were established using a new generalized Raina's fractional integral. Also from a suitable choices for the involved parameters in the proposed Definition 3.1 the Raina fractional integral operator, Hadamard fractional integral, Katugampola fractional integral, Prabhakar, Salim and Khan fractional integrals were deduced. As it was be remarked (Remarks 3.4,3.6,3.8,3.10,3.12 and 3.14), the main results remain valid for all the aforemetioned fractional integrals. In addition, some observations were presented to establish the generalization of the results obtained with respect to others previously found by L. Bougoffa [7] and S.S. Dragomir [13].

It should be noted that the results obtained can be generalized using the following fractional integral defined by T. Tunç et.al. in [36]

$$\left(\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;w}\varphi\right)(x) = \int_a^x \frac{g'(t)}{((g(x)-g(t))^{1-\frac{\lambda}{k}}}\mathcal{F}^{\sigma,k}_{\rho,\lambda}\left[w(x-t)^{\rho}\right]\varphi(t)dt, \quad (x > a)$$

and

$$\left(\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,b-;w}\varphi\right)(x) = \int_x^b \frac{g'(t)}{((g(x)-g(t))^{1-\frac{\lambda}{k}}}\mathcal{F}^{\sigma,k}_{\rho,\lambda}\left[w(t-x)^{\rho}\right]\varphi(t)dt, \quad (x < b),$$

where

$$\mathcal{F}^{\sigma,k}_{\rho,\lambda}(x) = \sum_{m=0}^{\infty}\frac{\sigma(m)}{k\Gamma_k(\rho k m + \lambda)}x^m,$$

$k > 0$, $g : [a, b] \to \mathbb{R}$ is a positive increasing function with a continuous derivative g' on $(a, b), \rho, \lambda > 0, |x| < \mathbb{R}$, $\sigma = (\sigma(1), .., \sigma(k), ..)$ is a bounded sequence of real numbers, Γ_k is the $k-$Gamma function, and φ is a function such that the integrals exist.

The authors hopes that the results will serve as a stimulus for future research in the area.

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