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Nota sobre algunas propiedades algebraicas de matrices obtenidas a partir de progresiones

A note on some algebraic properties of matrices obtained from progressions

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Resumen

En esta nota se introducen algunas matrices cuyas entradas usan progresiones aritméticas o geométricas para su formación. Estas matrices aparecen en matemática recreativa (cuadrados mágicos), en criptografía y en ecuaciones diferenciales. Se definen operaciones sobre el conjunto de estas matrices y se prueba que los conjuntos de estas matrices junto con estas operaciones tienen estructura de grupo, anillo y espacio vectorial.

Palabras claves:
Estructuras algebraicas, grupos y anillos, enseñanza del álgebra.
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Abstract

This note introduces some matrices whose entries use arithmetic or geometric progressions for their formation. These matrices appear in recreational mathematics (magic squares), cryptography and differential equations. On these matrices some operations were defined and it could be proved that together (matrices and operations), they have a group, ring, and vector space structure.

Keywords:
Algebraic structures, groups and rings, teaching of algebra.
2010 MSC: Primary: 97D40; Secundary: 9701

1. Introducction

In a previous work [FCA], the authors interested in addressing the notion of group in the basic courses of Abstract Algebra [F, H], introduce matrices whose entries use arithmetic or geometric progressions and check, under different operations, whether these sets comply with the axioms to be a group or not. In addition, they make observations on how to approach the concept of group in university courses that introduce Abstract Algebra.

The authors use matrices and progressions to work on the topic, since they consider that these are notions that the student has handled in previous courses and that do not involve a high level of complexity, seeking in this way, to introduce the concepts with elements already known to the students.

In this work, we show that the sets of matrices whose entries use arithmetic progressions have a ring structure and vector space, while the set of matrices whose entries use geometric progressions has a group structure.

In recreational math, magic squares feature prominently [An, R]. The matrices that are introduced in this work and that use arithmetic progressions correspond in the case of $M_n(1,1,n)$ to a normal magic square and in the case of $M_n(a,b,c)$ to a non-normal magic square. In the case of differential equations, the matrices worked here give rise to new results in differential algebra [AR].

2. Matrices and progressions

Definition 2.1. *Let us consider the following matrices of order n*

$$\begin{pmatrix} a & a+b & a+2b & \cdots & a+(n-1)b \\ a+c & a+b+c & a+2b+c & \cdots & a+(n-1)b+c \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a+(n-1)c & a+b+(n-1)c & a+2b+(n-1)c & \cdots & a+(n-1)b+(n-1)c \end{pmatrix}$$

whose rows and columns follow arithmetic progressions, with a as the element in position $(1,1)$ *and with* a,b,c *complex numbers. These matrices will noted as* $M_n(a,b,c)$ *since these elements characterize them completely. We note the set formed by these matrices as* MC_n*, that is,* $MC_n = \{M_n(a,b,c) | a,b,c \in \mathbb{C}\}$*. On some occasions, we will work with matrices of* MC_n *with* $a,b,c \in \mathbb{C}^*$*, forming a set that we will note as* MC_n^{*n}.

Examples: 1. Consider the matrix $M_3(1,-1,3)$ such that

$$M_3(1,-1,3) = \begin{pmatrix} 1 & 0 & -1 \\ 4 & 3 & 2 \\ 7 & 6 & 5 \end{pmatrix}$$

2. In the case that $a = 1$, $b = 1$ y $c = n$ we have that the matrix $M_n(1,1,n)$ is the list of the first n^2 natural numbers whose first row are the first n natural numbers, the second the natural numbers from $n+1$ up to $2n$ and so on. The elements of this matriz are the elements of a normal magic square of size $n \times n$. [An, R]
3. In the case that $a,b,c \in \mathbb{N}$, the elements of the matrix $M_n(a,b,c)$ form a non-normal magic square of siza $n \times n$. [An, R]
4. If $n \in \mathbb{N}$ then the matrix $M_n(1,1,n)$ is known as natural matrix [RG] and if $a,b,c \in \mathbb{N}$ then the matrix $M_n(a,b,c)$ is known as modified natural matrix.
5. Observe that the transpose of the matrix $M_n(a,b,c)$ is the matrix $M_n(a,c,b)$.
6. In the case in which $n = 7$, $a = 1$, $b = 1$ and $c = 7$ the matrix $M_7(1,1,7)$ is called a calendar matrix

because of its resemblance to a calendar.
7. The $M_{20\times 13}(1,7,1)$ module 13 matrix is known as the Tzolkin Synchronary -the count of days- in the Mayan culture [V].

Since the matrices $M_n(a,b,c)$ follow arithmetic progressions in both rows and columns, we have that the position i,j of these matrices is given by $a+(j-1)b+(i-1)c$. So $M_n(a,b,c)=(m_{ij})$, where $m_{ij} = a+(j-1)b+(i-1)c$, with $i,j=1,2,\ldots,n$, n positive integer and $a,b,c\in\mathbb{C}$.

Consider the matrix $M_n(e,f,h)=(n_{ij})$, where $n_{ij}=e+(j-1)f+(i-1)h$, with $i,j=1,2,\ldots,n$, n positive integer and $e,f,h\in\mathbb{C}$. We have that $M_n(a,b,c)+M_n(e,f,h)=(m_{ij})+(n_{ij})=(m_{ij}+n_{ij})=(a+(j-1)b+(i-1)c+e+(j-1)f+(i-1)h)=((a+e)+(j-1)(b+f)+(i-1)(c+h))=M_n(a+e,b+f,c+h)$. It is easy to prove that the matrix $M_n(0,0,0)$ is the neutral element of MC_n, and for the matrix $M_n(a,b,c)$ we have the matrix $M_n(-a,-b,-c)=-M_n(a,b,c)$ as an inverse element. The associativity and commutativity follow from the usual sum of matrices of order n.

From the above we have the following:

Proposition 2.1. *The matrices $M_n(a,b,c)$, with n positive integer and with the usual sum of matrices is an abelian group with neutral element $M_n(0,0,0)$, and the matrix $M_n(a,b,c)$ has the matrix $M_n(-a,-b,-c)=-M_n(a,b,c)$ as an inverse element.*

Let us now consider the matrices $M_n(a,b,c)=(m_{ij})$ where $m_{ij}=a+(j-1)b+(i-1)c$, $M_n(d,e,f)=(n_{ij})$ where $n_{ij}=d+(j-1)e+(i-1)f$ y $M_n(h,r,s)=(p_{ij})$ where $p_{ij}=h+(j-1)r+(i-1)s$.
We define the product $\times$ over the set of matrices MC_n, as follows: $M_n(a,b,c)\times M_n(d,e,f)=(m_{ij})\times(n_{ij})=(ad+(j-1)be+(i-1)cf)$, that is, $M_n(a,b,c)\times M_n(d,e,f)=M_n(ad,be,cf)$ and therefore, this product is a law of internal composition in MC_n.

For this product the matrix $M_n(1,1,1)$ is the identity element, a fact that follows easily from the definition of the product $\times$. It also follows easily from the definition that $M_n(a,b,c)\times M_n(d,e,f)=M_n(d,e,f)\times M_n(a,b,c)$. That is, the product $\times$ in MC_n satisfies the commutative property.

We will see next that the newly defined product satisfies the associative property on the set MC_n. Indeed,

$$\begin{aligned}(M_n(a,b,c)\times M_n(d,e,f))\times M_n(h,r,s) &= M_n(ad,be,cf)\times M_n(h,r,s)\\ &= M_n((ad)h,(be)r,(cf)s)=M_n(a(dh),b(er),c(fs))\\ &= M_n(a,b,c)\times M_n((dh),(er),(fs))\\ &= M_n(a,b,c)\times(M_n(d,e,f)\times M_n(h,r,s))\end{aligned}$$

We will see next that the defined product is compatible with the sum of matrices over the set of matrices MC_n. For this we calculate

$$\begin{aligned}(M_n(a,b,c)+M_n(d,e,f))\times M_n(h,r,s) &= M_n(a+d,b+e,c+f)\times M_n(h,r,s)\\ &= M_n((a+d)h,(b+e)r,(c+f)s)\\ &= M_n(ah+dh,br+er,cs+fs)\end{aligned}$$

On the other hand,

$$M_n(a,b,c) \times M_n(h,r,s) + M_n(d,e,f) \times M_n(h,r,s) = M_n(ah,br,cs) + M_n(dh,er,fs)$$
$$= M_n(ah+dh, br+er, cs+fs)$$

Comparing the two resulting matrices in each member and after applying the distributive and associative property of complex numbers, the equality of these two matrices is verified.

From the above we have the following:

Proposition 2.2. *The set MC_n with the sum + and the product $\times$ is an unitary commutative ring.*

It also follows that the invertible elements in this ring are the elements $M_n(a,b,c)$ where $a,b,c \in \mathbb{C}^*$.

The set MC_n is a vector space by defining a scalar multiplication as follows for $\alpha \in C$ y $M_n(a,b,c) \in MC_n$.

$$\alpha \cdot M_n(a,b,c) = \alpha \cdot (m_{ij}) = \alpha \cdot (a + (j-1)b + (i-1)c) = (\alpha a + (j-1)\alpha b + (i-1)\alpha c) = M_n(\alpha a, \alpha b, \alpha c)$$

Now, for the matrix $M_n(a,b,c)$ we have:

$$M_n(a,b,c) = a \cdot M_n(1,0,0) + b \cdot M_n(0,1,0) + c \cdot M_n(0,0,1)$$

This tells us that the given matrix is a linear combination of the matrices $M_n(1,0,0)$, $M_n(0,1,0)$, $M_n(0,0,1)$. In case of equating this combination of matrices to the null matrix we have that $a = 0$, $b = 0$ y $c = 0$. Therefore, we have thar the matrices $M_n(1,0,0)$, $M_n(0,1,0)$, $M_n(0,0,1)$ are linearly independent and generate the vector space MC_n, that is, they constitute a basis for this space. Thus, MC_n is a vector space of dimension 3.

Recall that in a ring R an ideal is a nomempty subset U of R, such that U is a subgroup of R under the operation of addition and for all u in U and r in R, both ur and ru are in U. I we consider sets of matrices of the form $M_n(a,0,0)$, $M_n(0,b,0)$, $M_n(0,0,c)$, $M_n(a,b,0)$, $M_n(a,0,c)$, $M_n(0,b,c)$ it follows easily that these sets with the operations of addition (+) and multiplication ($\times$) defined in the set MC_n are ideals of MC_n.

In the case of considering geometric progressions, we have the following definition:

Definition 2.2. *Let us consider the following matrices of order n*

$$\begin{pmatrix} a & ab & ab^2 & \cdots & ab^{n-1} \\ ac & abc & ab^2c & \cdots & ab^{n-1}bc \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ac^{n-1} & abc^{n-1} & ab^2c^{n-1} & \cdots & ab^{n-1}c^{n-1} \end{pmatrix},$$

whose rows and columns are geometric progressions, with a as the element in position $(1,1)$ *and with* a,b,c *complex numbers. We note these matrices by* $P_n(a,b,c)$ *since these elements characterize them completely. We will note the set formed by these matrices by* PC_n*, that is,* $PC_n = \{P_n(a,b,c) | a,b,c \in \mathbb{C}\}$*. On some occasions, we will work with matrices of* P_n *with* $a,b,c \in \mathbb{C}^*$*, forming a set that we will notice by* PC_n^**.*

Example: Consider the matrix $P_3(2,-1,3)$ such that

$$P_3(2,-1,3) = \begin{pmatrix} 2 & -2 & 2 \\ 6 & -6 & 6 \\ 18 & -18 & 18 \end{pmatrix}$$

Given that the matrices $P_n(a,b,c)$ follow geometric progressions in both rows and columns, we have that the position i, j of said matrices is given by $ab^{j-1}c^{i-1}$ with $i, j = 1, 2, \ldots, n$, n positive integer and $a, b, c \in \mathbb{C}^*$. Once we have an expression for the matrices $P_n(a,b,c)$ we will prove that the operation * is a law of internal composition on the set PC_n^*.

Consider the matrix $P_n(e, f, h) = (n_{ij})$, where $n_{ij} = ef^{j-1}h^{i-1}$, with $i, j = 1, 2, \ldots, n$, n positive integer and $a, b, c \in \mathbb{C}^*$. We have that $P_n(a,b,c) * P_n(e, f, h) = (m_{ij}) * (n_{ij}) = (m_{ij}n_{ij}) = (ab^{j-1}c^{i-1}ef^{j-1}h^{i-1}) = ((ae)(bf)^{j-1}(ch)^{i-1}) = P_n(ae, bf, ch)$.

It is easy to prove that the matrix $P_n(1,1,1)$ is the neutral element of P_n, and for the matrix $P_n(a,b,c)$ with $a, b, c \in \mathbb{C}^*$ we have the matrix $P_n^{-1}(a,b,c) = P_n(a^{-1}, b^{-1}, c^{-1})$ as an inverse element. The associativity and commutativity of the matrices $P_n(a,b,c)$ in P_n^* follow from the associativity and commutativity of the product of the complex numbers.

From the above we have the following:

Proposition 2.3. *The matrices $P_n(a,b,c) = (n_{ij})$ where $n_{ij} = ab^{j-1}c^{i-1}$ with $i, j = 1, 2, \ldots, n$, n positive integer and $a, b, c \in \mathbb{C}^*$, with the product of matrices $*$ is an abelian group with neutral element $P_n(1,1,1)$, and for the matrix $P_n(a,b,c)$ we have $P_n^{-1}(a,b,c) = P_n(a^{-1}, b^{-1}, c^{-1})$ as an inverse element.*

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