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Nuevas desigualdades tipo Jensen para funciones φ-convexas.

New Jensen inequalities for φ-convex functions.

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Resumen

La desigualdad integral de Jensen tiene mucha importancia en cuanto a sus aplicaciones en diferentes campos de las matemáticas. En este artículo encontramos una nueva desigualdad tipo Jensen para funciones cuya segunda derivada en valor absoluto es φ-convexa.

Palabras claves: funciones φ-convexas, funciones convexas, función de Green, desigualdades del tipo Jensen.

Abstract

Jensen integral inequality has got much importance regarding their applications in different fields of mathematics. In this paper we find a new bound for the Jensen gap for functions whose double derivatives in absolute function are φ-convex.

Keywords:
φ-convex function, convex function, Green function, Jensen type inequalities.

1. Introducción

The convex functions play a significant role in many fields, for example in biological system, economy, optimization and so on [7, 18]. And many important inequalities are established for these class of functions. Also the evolution of the concept of convexity has had a great impact in the community of investigators. In recent years, for example, generalized concepts such as s-convexity (see[2]), h-convexity (see [19, 21]), φ-convexity (see [24, 27]), and others, as well as combinations of these new concepts have been introduced.

In the following theorem, Jensen integral inequality has been presented [12].

Theorem 1.1. *([12]) Let $[a,b] \subset \mathbb{R}$ and $f, g : [c,d] \to \mathbb{R}$ be two measurable functions such that $f(t) \in [a,b]$, $\forall t \in [c,d]$. Let the function $\Psi : [a,b] \to \mathbb{R}$ be convex and $g, fg, (\Psi \circ f) \cdot g$ are integrable functions on $[c,d]$. Also suppose that $g(t) \geq 0$ for all $t \in [c,d]$ and $\int_c^d g(t)dt > 0$, then*

$$\Psi\left(\frac{\int_c^d f(t)g(t)dt}{\int_c^d g(t)dt}\right) \leq \frac{\int_c^d (\Psi \circ f)(t)g(t)dt}{\int_c^d g(t)dt}. \tag{1}$$

We will deduce other classical inequalities from (1) for φ-convex functions. We organize the remaining paper as: In Section 2, we present a new main result following by a remark and one numerical experiment, in Section 3, we present a new result applying a divergencia known as Csiszár divergence which completes the section.

2. Main Results

To start the main results, we need the following Green function defined on $[\beta_1, \beta_2] \times [\beta_1, \beta_2]$

$$\mathcal{G}(x,y) = \begin{cases} \frac{(x-\beta_2)(y-\beta_1)}{\beta_2-\beta_1} & \text{if} \quad \beta_1 \leq y \leq x \\ \frac{(y-\beta_2)(x-\beta_1)}{\beta_2-\beta_1} & \text{if} \quad x \leq y \leq \beta_2. \end{cases} \tag{2}$$

This is a convex function with respect to both the variables x and y.

Lemma 2.1 ([11]). *The following identity for $\mathcal{L} \in C^2[\beta_1, \beta_2]$ hold.*

$$\mathcal{L}(x) = \frac{\beta_2 - x}{\beta_2 - \beta_1}\mathcal{L}(\beta_1) + \frac{x - \beta_1}{\beta_2 - \beta_1}\mathcal{L}(\beta_2) + \int_{\beta_1}^{\beta_2} \mathcal{G}(x,y)\mathcal{L}''(y)dy,$$

where $\mathcal{G}$ is related to the Green function (2).

In the following theorem, we present a new bound for the Jensen gap by using functions whose double derivatives in the absolute function are φ-convex.

Theorem 2.2. *Let $\mathcal{L} \in C^2[\beta_1, \beta_2]$ be a function such that $|\mathcal{L}''|$ is φ-convex. Let h, f be two real valued functions defined on $[c,d]$ such that $h(z) \in [\beta_1, \beta_2]$ for all $z \in [c,d]$ with $f, hf, (\mathcal{L} \circ h)f$ as integrable functions on $[c,d]$. Also suppose that $f(z) \geq 0$ on $[c,d]$ with $\int_c^d f(z)dz = K > 0$, then*

$$\begin{aligned} &\left|\frac{1}{K}\int_c^d (\mathcal{L} \circ h)(z)f(z)dz - \mathcal{L}\left(\frac{1}{K}\int_c^d h(z)f(z)dz\right)\right| \\ &\leq \frac{|\mathcal{L}''(\beta_2)|}{2}\left[\frac{1}{K}\int_c^d f(z)h^2(z)dz - \left(\frac{1}{K}\int_c^d f(z)h(z)dz\right)^2\right] \\ &+ \frac{\varphi(|\mathcal{L}''(\beta_1)|, |\mathcal{L}''(\beta_2)|)}{\beta_2 - \beta_1}\left[\left(\frac{1}{\sqrt[3]{6}K}\int_c^d f(z)h(z)dz\right)^3 - \frac{1}{6K}\int_c^d f(z)h^3(z)dz\right. \\ &\left. + \frac{\beta_2}{2K}\int_c^d f(z)h^2(z)dz - \left(\frac{\sqrt{\beta_2}}{\sqrt{2}K}\int_c^d f(z)h(z)dz\right)^2\right]. \end{aligned} \tag{3}$$

Proof. Using Lemma 2.1

$$\begin{aligned}\frac{1}{K}\int_c^d (\mathcal{L}\circ h)(z)f(z)dz &= \frac{1}{K}\int_c^d \left[\frac{\beta_2-h(z)}{\beta_2-\beta_1}\mathcal{L}(\beta_1)+\frac{h(z)-\beta_1}{\beta_2-\beta_1}\mathcal{L}(\beta_2)\right.\\ &+\left.\int_{\beta_1}^{\beta_2}\mathcal{G}(h(z),y)\mathcal{L}''(y)dy\right]f(z)dz\end{aligned}\tag{4}$$

and

$$\begin{aligned}\mathcal{L}\left(\frac{1}{K}\int_c^d h(z)f(z)dz\right) &= \frac{\beta_2-\frac{1}{K}\int_c^d h(z)f(z)dz}{\beta_2-\beta_1}\mathcal{L}(\beta_1)+\frac{\frac{1}{K}\int_c^d h(z)f(z)dz-\beta_1}{\beta_2-\beta_1}\mathcal{L}(\beta_2)\\ &+\int_{\beta_1}^{\beta_2}\mathcal{G}\left(\frac{1}{K}\int_c^d h(z)f(z)dz, y\right)\mathcal{L}''(y)dy\end{aligned}\tag{5}$$

Subtracting (4) from (5), we obtain the following result

$$\begin{aligned}\frac{1}{K}\int_c^d (\mathcal{L}\circ h)(z)f(z)dz-\mathcal{L}\left(\frac{1}{K}\int_c^d h(z)f(z)dz\right) &= \int_{\beta_1}^{\beta_2}\left[\frac{1}{K}\int_c^d \mathcal{G}(h(z),y)f(z)dz\right.\\ &-\left.\mathcal{G}\left(\frac{1}{K}\int_c^d h(z)f(z)dz, y\right)\right]\mathcal{L}''(y)dy.\end{aligned}\tag{6}$$

Taking absolute of (6) and using the change of variable $y = t\beta_1 + (1-t)\beta_2$ for $t \in [0,1]$, we get

$$\begin{aligned}&\left|\frac{1}{K}\int_c^d (\mathcal{L}\circ h)(z)f(z)dz-\mathcal{L}\left(\frac{1}{K}\int_c^d h(z)f(z)dz\right)\right|\\ &\le \int_{\beta_1}^{\beta_2}\left|\frac{1}{K}\int_c^d \mathcal{G}(h(z),y)f(z)dz-\mathcal{G}\left(\frac{1}{K}\int_c^d h(z)f(z)dz, y\right)\right||\mathcal{L}''(y)|dy\\ &= (\beta_2-\beta_1)\int_0^1\left[\frac{1}{K}\int_c^d \mathcal{G}(h(z), t\beta_1+(1-t)\beta_2)f(z)dz\right.\\ &-\left.\mathcal{G}\left(M, t\beta_1+(1-t)\beta_2\right)\right]|\mathcal{L}''(t\beta_1+(1-t)\beta_2)|dt,\end{aligned}\tag{7}$$

where $M = \frac{1}{K}\int_c^d h(z)f(z)dz$.

Since $|\mathcal{L}''|$ is a φ-convex function, therefore (7) takes the form

$$\begin{aligned}
&\left|\frac{1}{K}\int_c^d (\mathcal{L}\circ h)(z)f(z)dz - \mathcal{L}\left(\frac{1}{K}\int_c^d h(z)f(z)dz\right)\right|\\
&\leq (\beta_2-\beta_1)\int_0^1\left[\frac{1}{K}\int_c^d \mathcal{G}(h(z),t\beta_1+(1-t)\beta_2)f(z)dz\right.\\
&\left.-\mathcal{G}\left(M,t\beta_1+(1-t)\beta_2\right)\right](|\mathcal{L}''(\beta_2)|+t\varphi(|\mathcal{L}''(\beta_1)|,|\mathcal{L}''(\beta_2)|))\,dt\\
&=(\beta_2-\beta_1)\bigg(\int_0^1\frac{|\mathcal{L}''(\beta_2)|}{K}\int_c^d \mathcal{G}(h(z),t\beta_1+(1-t)\beta_2)f(z)dzdt\\
&+\int_0^1\frac{t\varphi(|\mathcal{L}''(\beta_1)|,|\mathcal{L}''(\beta_2)|)}{K}\int_c^d \mathcal{G}(h(z),t\beta_1+(1-t)\beta_2)f(z)dzdt\\
&-\int_0^1|\mathcal{L}''(\beta_2)|\mathcal{G}\left(M,t\beta_1+(1-t)\beta_2\right)dt-\int_0^1 t\varphi(|\mathcal{L}''(\beta_1)|,|\mathcal{L}''(\beta_2)|)\mathcal{G}\left(M,t\beta_1+(1-t)\beta_2\right)dt\bigg)\\
&=(\beta_2-\beta_1)\bigg(\frac{|\mathcal{L}''(\beta_2)|}{K}\int_c^d f(z)\Big(\int_0^1\mathcal{G}(h(z),t\beta_1+(1-t)\beta_2)dt\Big)dz\\
&+\frac{\varphi(|\mathcal{L}''(\beta_1)|,|\mathcal{L}''(\beta_2)|)}{K}\int_c^d f(z)\Big(\int_0^1 t\mathcal{G}(h(z),t\beta_1+(1-t)\beta_2)dt\Big)dz\\
&-|\mathcal{L}''(\beta_2)|\int_0^1\mathcal{G}\left(M,t\beta_1+(1-t)\beta_2\right)dt\\
&-\varphi(|\mathcal{L}''(\beta_1)|,|\mathcal{L}''(\beta_2)|)\int_0^1 t\mathcal{G}\left(M,t\beta_1+(1-t)\beta_2\right)dt\bigg),
\end{aligned}\tag{8}$$

we note that

$$\begin{aligned}
&\int_0^1 t\mathcal{G}(h(z),t\beta_1+(1-t)\beta_2)dt=\frac{1}{(\beta_2-\beta_1)^3}\Big[\frac{\beta_1h^3(z)}{6}-\frac{\beta_2h^3(z)}{6}+\frac{\beta_2^2h^2(z)}{2}-\frac{\beta_2\beta_1h^2(z)}{2}\\
&-\frac{\beta_2^3h(z)}{3}-\frac{\beta_1^3h(z)}{6}+\frac{\beta_2\beta_1^2h(z)}{2}+\frac{\beta_2\beta_1^3}{6}-\frac{\beta_2^2\beta_1^2}{2}+\frac{\beta_2^3\beta_1}{3}\Big],
\end{aligned}\tag{9}$$

$$\begin{aligned}
&\int_0^1 t\mathcal{G}(M,t\beta_1+(1-t)\beta_2)dt=\frac{1}{(\beta_2-\beta_1)^3}\Big[\frac{\beta_1M^3}{6}-\frac{\beta_2M^3}{6}+\frac{\beta_2^2M^2}{2}-\frac{\beta_2\beta_1M^2}{2}\\
&-\frac{\beta_2^3M}{3}-\frac{\beta_1^3M}{6}+\frac{\beta_2\beta_1^2M}{2}+\frac{\beta_2\beta_1^3}{6}-\frac{\beta_2^2\beta_1^2}{2}+\frac{\beta_2^3\beta_1}{3}\Big],
\end{aligned}\tag{10}$$

$$\int_0^1\mathcal{G}(h(z),t\beta_1+(1-t)\beta_2)dt=\frac{h^2(z)-\beta_1h(z)-\beta_2h(z)+\beta_1\beta_2}{2(\beta_2-\beta_1)},\tag{11}$$

$$\int_0^1\mathcal{G}(M,t\beta_1+(1-t)\beta_2)dt=\frac{M^2-\beta_1M-\beta_2M+\beta_1\beta_2}{2(\beta_2-\beta_1)},\tag{12}$$

Substituting the values from (9)–(12) in (8) and simplifying, we get

$$\begin{aligned}
&\left|\frac{1}{K}\int_c^d (\mathcal{L}\circ h)(z)f(z)dz - \mathcal{L}\left(\frac{1}{K}\int_c^d h(z)f(z)dz\right)\right| \\
&\leq \frac{|\mathcal{L}''(\beta_2)|}{2}\left[\frac{1}{K}\int_c^d f(z)h^2(z)dz - \left(\frac{1}{K}\int_c^d f(z)h(z)dz\right)^2\right] \\
&+ \frac{\varphi(|\mathcal{L}''(\beta_1)|, |\mathcal{L}''(\beta_2)|)}{\beta_2-\beta_1}\left[\left(\frac{1}{\sqrt[3]{6}K}\int_c^d f(z)h(z)dz\right)^3 - \frac{1}{6K}\int_c^d f(z)h^3(z)dz\right. \\
&\left. + \frac{\beta_2}{2K}\int_c^d f(z)h^2(z)dz - \left(\frac{\sqrt{\beta_2}}{\sqrt{2}K}\int_c^d f(z)h(z)dz\right)^2\right].
\end{aligned}$$

■

Remark 2.3. *Note that if $\varphi(x,y) = x - y$ in the Theorem 2.2 we obtain the Theorem 2 in [1].*

Now we demonstrate some numerical experiments to show the tightness of the bound (3)

Example 2.4. *Let $\mathcal{L}(z) = \frac{z^5}{20}, h(z) = z, f(z) = 1$, for all $z \in [0,1]$, $\mathcal{L}''(z) = z^3$ is φ-convex function with $\varphi(x,y) = 3y^2(x-y) + 3y(x-y)^2 + (x-y)^3$. Also $h(z) \in [0,1]$ for all $z \in [0,1]$, therefore using inequality (3) for these facts with $[\beta_1,\beta_2] = [c,d] = [0,1]$, we obtain $\left|\frac{1}{K}\int_c^d (\mathcal{L}\circ h)(z)f(z)dz - \mathcal{L}\left(\frac{1}{K}\int_c^d h(z)f(z)dz\right)\right| =$ $0,0067708$ and corresponding right hand side gives $0,02083$ Thus from inequality (3) we conclude that*

$$0,0067708 < 0,02083$$

3. Applications

CsiszÃ¡r [11] introduced a divergence known as CsiszÃ¡r divergence, which is the base for other divergences for example Kullback-Leibler divergence, Jeffrey's divergence etc. Divergences have many applications in various fields of Science, Technology, Genetics [3], Applied Statistics [8] etc. Jensen inequality plays a vital role to deduce the estimates for various divergences [4], [5], [9], [16]. In this section, we present some applications of our result for CsiszÃ¡r divergence.

Definition 3.1. *Let $[\beta_1,\beta_2] \subseteq \mathbb{R}$ and $g : [\beta_1,\beta_2] \to \mathbb{R}$ be a function. Also let $V : [c,d] \to [\beta_1,\beta_2]$, $W : [c,d] \to (0,\infty)$ be two functions such that $\frac{V(z)}{W(z)} \in [\beta_1,\beta_2]$ for all $z \in [c,d]$ then the CsiszÃ¡r divergence is defined by [11]*

$$D^c(V,W) = \int_c^d W(z)g\left(\frac{V(z)}{W(z)}\right)dz.$$

Theorem 3.2. *Let $g \in C^2[\beta_1,\beta_2]$ be a function such that $|g''|$ is φ-convex. Also $V : [c,d] \to [\beta_1,\beta_2]$, W :*

$[c,d] \to (0,\infty)$ *be two functions such that* $\frac{\int_c^d V(z)dz}{\int_c^d W(z)dz}, \frac{V(z)}{W(z)} \in [\beta_1, \beta_2]$*, for all* $z \in [c,d]$ *then*

$$\left|\frac{1}{\int_c^d W(z)dz} D^c(V,W) - g\left(\frac{\int_c^d V(z)dz}{\int_c^d W(z)dz}\right)\right|$$
$$\leq \frac{g''(\beta_2)|}{2}\left[\frac{\int_c^d \frac{V^2(z)}{W(z)}dz}{\int_c^d W(z)dz} - \left(\frac{\int_c^d V(z)dz}{\int_c^d W(z)dz}\right)^2\right]$$
$$+ \frac{\varphi(|g''(\beta_1)|, |g''(\beta_2)|)}{\beta_2 - \beta_1}\left[\left(\frac{1}{\sqrt[3]{6}}\frac{\int_c^d V(z)dz}{\int_c^d W(z)dz}\right)^3 - \frac{1}{6}\frac{\int_c^d \frac{V^3(z)}{W^2(z)}dz}{\int_c^d W(z)dz}\right. \tag{13}$$
$$\left.+ \frac{\beta_2}{2}\frac{\int_c^d \frac{V^2(z)}{W(z)}dz}{\int_c^d W(z)dz} - \left(\frac{\sqrt{\beta_2}}{\sqrt{2}}\frac{\int_c^d V(z)dz}{\int_c^d W(z)dz}\right)^2\right].$$

Proof. Using (3) for $\mathcal{L} = g,\ h(z) = \frac{V(z)}{W(z)}$ and $f(z) = W(z)$, we obtain (13). ■

4. Conclusions

In this paper we have established Jensen type inequalities given by Adil Khan M, Shahid Khan, Yu-Ming Chu in [1] for the case φ-convex functions. We expect that the ideas and techniques used in this paper may inspire interested readers to explore some new applications of these newly introduced explore some new applications of these newly introduced functions in various fields of pure and applied sciences.

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