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On generalized η-convex functions and the related inequalities

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Abstract

The concept of η-convex function was recently introduced by Gordji et al. [7]. A function $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be η-convex with respect to a function $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if

$$f(tx + (1 - t)y) \leq f(y) + t\eta(f(x), f(y)),$$

for all $x, y \in I$ and $t \in [0, 1]$. In this paper, we introduce and study a generalization of η-convex functions using the fractal calculus developed by Yang [10]. Among other results, we show that this class of functions satisfy some Hermite-Hadamard and Fejér type inequalities.

Keywords: convex function, η-convex function, fractal set, generalized η-convex function.
MSC 2010. Primary 26D07, 26D15; Secondary 26A51, 39B62.

Resumen

El concepto de función η-convexa fue recientemente introducido por Gordji et al. [7]. Una función $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ se dice η-convexa con respecto a una función $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, si

$$f(tx + (1 - t)y) \leq f(y) + t\eta(f(x), f(y)),$$

for all $x, y \in I$ and $t \in [0, 1]$. En este trabajo, introducimos y estudiamos una generalización de las funciones η-convexas usando el cálculo fractal desarrollado por Yang [10]. Entre otros resultados, mostramos que este tipo de funciones satisfacen algunas desigualdades del tipo Hermite-Hadamard y del tipo Fejér.

Palabras claves: función convexa, función η-convexa, conjunto fractal, función η-convexa generalizada.

1. Introduction and preliminaries

Convex sets, convex functions and their generalizations are important in applied mathematics, especially in nonlinear programming and optimization theory. For example in economics, convexity plays a fundamental role in equilibrium and duality theory. The convexity of sets and functions has been the subject of many studies in recent years, among which we can mention several generalized variants; as done by Gordji et al. [7] when they introduced the class of η-convex functions and the other definitions as follows.

Definición 1.1. *A function $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be η-convex with respect to η: $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if*

$$f(tx + (1 - t)y) \leq f(y) + t\eta(f(x), f(y)),$$

for all $x, y \in I$ and $t \in [0, 1]$. On the other hand, f is called η-quasiconvex, if

$$f(tx + (1 - t)y) \leq max\{f(y), f(y) + \eta(f(x), f(y))\},$$

for all $x, y \in I$ and $t \in [0, 1]$. Moreover, f is called η-affine, if

$$f(tx + (1 - t)y) \leq f(y) + t\eta(f(x), f(y)),$$

for all $x, y, t \in \mathbb{R}$.

Observación 1.2. *In this work we use the notation given by Awan et al. [1] for η-convex functions.*

The concept of local fractional calculus (also called fractal calculus) has received considerable attention for its application in non-differentiable problems of science and engineering. In the sense of Mandelbrot, a fractal set is the one whose Hausdorff dimension strictly exceeds the topological dimension (see [5], [6] and [8]). Many researchers studied the properties of functions on fractal space and constructed many kinds of fractional calculus by using different approaches (see [2] , [3], [10] and [12]). In particular, Yang [10] stated the analysis of local fractional functions on fractal space systematically, which includes local fractional calculus and the monotonicity of functions. The theory of fractal sets developed by Yang establishes the following. For $0 < \alpha \leq 1$, we have the following α-type sets:

$\mathbb{Z}^\alpha = \{0^\alpha, \pm 1^\alpha, \pm 2^\alpha, ..., \pm n^\alpha, ...\}$ (integer numbers α-type).

$\mathbb{Q}^\alpha = \left\{m^\alpha = \left(\frac{p}{q}\right)^\alpha : p, q \in \mathbb{Z}, q \neq 0\right\}$ (rational numbers α-type).

$\mathbb{J}^\alpha = \left\{m^\alpha \neq \left(\frac{p}{q}\right)^\alpha : p, q \in \mathbb{Z}, q \neq 0\right\}$ (irrational numbers α-type).

$\mathbb{R}^\alpha = \mathbb{Q}^\alpha \cup \mathbb{J}^\alpha$ (real numbers α-type).

We call fractal set to $\mathbb{R}^\alpha$ and any subset of it. The following facts are found in [4], [10] and [11].

If a^α, b^α and c^α belong to the set $\mathbb{R}^\alpha$ of α-type real numbers, then we have the following properties:

1. $a^\alpha + b^\alpha$ and $a^\alpha b^\alpha$ belong to the set $\mathbb{R}^\alpha$.
2. $a^\alpha + b^\alpha = b^\alpha + a^\alpha = (a + b)^\alpha = (b + a)^\alpha$.
3. $a^\alpha + (b^\alpha + c^\alpha) = (a^\alpha + b^\alpha) + c^\alpha$.
4. $a^\alpha b^\alpha = b^\alpha a^\alpha = (ab)^\alpha = (ba)^\alpha$.
5. $a^\alpha (b^\alpha c^\alpha) = (a^\alpha b^\alpha) c^\alpha$.
6. $a^\alpha (b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$.
7. $a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha$ and $a^\alpha 1^\alpha = 1^\alpha a^\alpha = a^\alpha$.

It is important to note that in this theory the number $(a^2)^\alpha \in \mathbb{R}^\alpha$ will be represented by $a^{2\alpha}$.

Now we introduce some basic definitions about the local factional calculus.

Definición 1.3. *[10] A non-differentiable function $f : \mathbb{R} \to \mathbb{R}^\alpha$, $x \to f(x)$ is called local fractional continuous at x_0, if for any $\varepsilon > 0$, there exists $\delta > 0$, such that*

$$|f(x) - f(x_0)| < \varepsilon$$

holds for $|x - x_0| < \delta$, where $\varepsilon, \delta \in \mathbb{R}$. If a function f is local fractional continuous on an interval I, we denote $f \in C_\alpha(I)$.

Definición 1.4. *[10] The local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined by*

$$f^{(\alpha)}(x_0) = \left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha},$$

where $\Delta^\alpha (f(x) - f(x_0)) \cong \Gamma(1+\alpha)(f(x) - f(x_0))$ and Γ is the familiar Gamma function.

Let $f^{(\alpha)}(x) = D_x^\alpha f(x)$. If there exists $f^{((k+1)\alpha)}(x) = \overbrace{D_x^\alpha \cdots D_x^\alpha}^{k+1 \text{ times}} f(x)$ for any $x \in I \subseteq \mathbb{R}$, then we denote $f \in D_{(k+1)\alpha}(I)$, where $k = 0, 1, 2, \ldots$.

Definición 1.5. *[10] Let $f \in C_\alpha[a,b]$. The local fractional integral of f on the interval $[a,b]$ of order α (denoted by ${}_aI_b^{(\alpha)} f$) is defined by*

$${}_aI_b^{(\alpha)} f(t) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha,$$

with $\Delta t = \text{máx}\{\Delta t_0, \Delta t_1, \ldots, \Delta t_{N-1}\}$ and $\Delta t_j = t_{j+1} - t_j$ for $j = 0, 1, \ldots, N-1$, where $a = t_0 < t_1 < \cdots < t_i < \cdots < t_{N-1} < t_N = b$ is a partition of the interval $[a,b]$.

Here, it follows that ${}_aI_b^{(\alpha)} f = 0$ if $a = b$ and ${}_aI_b^{(\alpha)} f = -{}_bI_a^{(\alpha)} f$ if $a < b$. If ${}_aI_x^{(\alpha)} f$ there exits for any $x \in [a,b]$, then it is denoted by $f \in I_x^{(\alpha)}[a,b]$.

In 2014, H. Mo et al. [9] used the local fractional calculus to introduce the following generalized convex function.

Definición 1.6. *[9] Let $f : I \to \mathbb{R}^\alpha$. For any $x, y \in I$ and $t \in [0,1]$, if the following inequality*

$$f(tx + (1-t)y) \leq t^\alpha f(x) + (1-t)^\alpha f(y),$$

holds, then f is called a generalized convex function on I.

We will denote by $GC_\alpha(I)$ to the set of the generalized convex functions on I, that is to say,

$$GC_\alpha(I) = \{f : I \to \mathbb{R}^\alpha | f \text{ is a generalized convex function on } I\}.$$

2. Main results

In this section we introduce the definition of generalized η-convex function and study some relevant properties.

Definición 2.1. *A function $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}^\alpha$ is said to be generalized η-convex with respect to $\eta : \mathbb{R}^\alpha \times \mathbb{R}^\alpha \to \mathbb{R}^\alpha$, if*

$$f(tx + (1-t)y) \leq f(y) + t^\alpha \eta(f(x), f(y)),$$

for all $x, y \in I$ and $t \in [0,1]$. On the other, f is called generalized η-quasiconvex, if

$$f(tx + (1-t)y) \leq max\{f(y), f(y) + \eta(f(x), f(y))\},$$

for all $x, y \in I$ and $t \in [0,1]$. Moreover, f is called generalized η-affine, if

$$f(tx + (1-t)y) = f(y) + t^\alpha \eta(f(x), f(y)),$$

for all $x, y, t \in \mathbb{R}$.

The family of all generalized η-convex functions in an interval $I = [a,b]$ is denoted by $\eta\text{-}GC_\alpha(I)$; which is,

$$\eta\text{-}GC_\alpha(I) = \{f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}^\alpha | f \text{ is a generalized } \eta\text{-convex function}\}.$$

Observación 2.2. *Note that for particular cases of the number $0 < \alpha \leq 1$ and the function η, we recover well known classical concepts of convex functions as is shown below.*

(1) *If $\alpha = 1$ then generalized η-convex functions are η-convex functions.*
(2) *If $\eta(x^\alpha, y^\alpha) = x^\alpha - y^\alpha$ then generalized η-convex functions are generalized convex functions.*
(3) *If $\alpha = 1$ and $\eta(x^\alpha, y^\alpha) = x^\alpha - y^\alpha$ then generalized η-convex functions are convex functions.*

If $f \in \eta\text{-}GC_\alpha(I)$ and $x = y$ then $f(x) \leq f(x) + t^\alpha \eta(f(x), f(x))$; that is, $0^\alpha \leq t^\alpha \eta(f(x), f(x))$ and hence, $0^\alpha \leq \eta(f(x), f(x))$. Also, if $f \in \eta\text{-}GC_\alpha(I)$ and $t = 1$ then $f(x) \leq f(y) + \eta(f(x), f(y))$, which implies that $f(x) - f(y) \leq \eta(f(x), f(y))$. On the other hand, if $f \in GC_\alpha(I)$ and $\eta : \mathbb{R}^\alpha \times \mathbb{R}^\alpha \to \mathbb{R}^\alpha$ is a function which satisfies the equality $\eta(a^\alpha, b^\alpha) \geq a^\alpha - b^\alpha$ for all $a^\alpha, b^\alpha \in \mathbb{R}^\alpha$, then

$$\begin{aligned} f(tx + (1-t)y) &\leq t^\alpha f(x) + (1-t)^\alpha f(y) = f(y) + t^\alpha [f(x) - f(y)] \\ &\leq f(y) + t^\alpha \eta(f(x), f(y)). \end{aligned}$$

This shows that $f \in \eta\text{-}GC_\alpha(I)$.

Ejemplo 2.3. For a function $f \in GC_\alpha(I)$, we may find another function η other than the function $\eta(x^\alpha, y^\alpha) = x^\alpha - y^\alpha$ such that $f \in \eta\text{-}GC_\alpha(I)$. Consider the function $f \in GC_\alpha(I)$ defined by $f(x) = x^{2\alpha}$ and $\eta(x^\alpha, y^\alpha) = 2^\alpha x^\alpha + y^\alpha$. Then, we have

$$\begin{aligned} f(tx + (1-t)y) &= (tx + (1-t)y)^{2\alpha} \leq (y^2 + tx^2 + t(1-t)2xy)^\alpha \\ &\leq (y^2 + tx^2 + t(1-t)(x^2 + y^2))^\alpha \leq (y^2 + t(x^2 + x^2 + y^2))^\alpha \\ &= y^{2^\alpha} + t^\alpha (2^\alpha x^{2\alpha} + y^{2\alpha}) \leq f(y) + t^\alpha \eta(f(x), f(y)) \end{aligned}$$

for all $x, y \in \mathbb{R}$ and $t \in (0,1)$. Since $x^{2\alpha} \leq y^{2^\alpha} + (2^\alpha x^{2\alpha} + y^{2\alpha})$ and $y^{2\alpha} \leq y^{2\alpha}$ for all $x, y \in I$, then the inequality also is true for $t = 1$ and $t = 0$, respectively. This shows that $f \in \eta\text{-}GC_\alpha(I)$. Note that the function $f(x) = x^{2\alpha}$ is generalized η-convex with respect to every $\eta(x^\alpha, y^\alpha) = a^\alpha x^\alpha + b^\alpha y^\alpha$ with $a^\alpha \geq 1^\alpha$, $b^\alpha \geq -1^\alpha$ and $x, y \in \mathbb{R}$.

Definición 2.4. *A function* $\eta : \mathbb{R}^\alpha \times \mathbb{R}^\alpha \to \mathbb{R}^\alpha$ *is called:*

(1) *Nonnegatively homogeneous, if* $\eta(\lambda^\alpha x^\alpha, \lambda^\alpha y^\alpha) = \lambda^\alpha \eta(x^\alpha, y^\alpha)$ *for all* $x^\alpha, y^\alpha \in \mathbb{R}^\alpha$ *and* $\lambda^\alpha \geq 0^\alpha$.
(2) *Additive, if* $\eta(x_1^\alpha, y_1^\alpha) + \eta(x_2^\alpha, y_2^\alpha) = \eta(x_1^\alpha + x_2^\alpha, y_1^\alpha + y_2^\alpha)$ *for all* $x_1^\alpha, x_2^\alpha, y_1^\alpha, y_2^\alpha \in \mathbb{R}^\alpha$.
(3) *Nonnegatively linear, if satisfies conditions* (1) *and* (2).

The proofs of the following three theorems are straightforward and are thus omitted.

Teorema 2.5. *Consider a function* $f \in \eta\text{-}GC_\alpha(I)$ *such that* η *is nonnegatively homogeneous. Then, for any* $\lambda^\alpha \geq 0^\alpha$, $\lambda^\alpha f \in \eta\text{-}GC_\alpha(I)$.

Teorema 2.6. *Let* $f, g \in \eta\text{-}GC_\alpha(I)$ *such that* η *is additive. Then* $f + g \in \eta\text{-}GC_\alpha(I)$.

Teorema 2.7. *Let* $f_i \in \eta\text{-}GC_\alpha(I)$ *for* $i = 1, ..., n$, *such that* η *is nonnegatively linear. Then, for* $\lambda_i^\alpha \geq 0^\alpha$, $i = 1, ..., n$, *the function* $f = \sum_{i=1}^{n} \lambda_i^\alpha f_i : I \to \mathbb{R}^\alpha \in \eta\text{-}GC_\alpha(I)$.

Next we introduce a generalized notion of a affine function on fractal sets and establish a result that relates a generalized affine function with a η-affine function.

Definición 2.8. *A function* $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}^\alpha$ *is said to be generalized affine, if for each pair* $x, y \in \mathbb{R}$ *and each pair* p, q *with* $p + q = 1$, *we have*

$$f(px + qy) = p^\alpha f(x) + q^\alpha f(y).$$

Teorema 2.9. *For a function* $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}^\alpha$ *the following statements are equivalent:*

(1) f *is a generalized affine function.*
(2) f *is a generalized* η*-affine for each* η.

Demostración. The implication (1) $\Rightarrow$ (2) is obvious. To show (2) $\Rightarrow$ (1) we consider $t \in \mathbb{R}$. Then $f(t) = f(t \cdot 1 + (1 - t) \cdot 0) = f(0) + t^\alpha \eta(f(1), f(0))$. Now, for $x, y \in \mathbb{R}$, we have

$$\begin{aligned} f(tx + (1 - t)y) &= f(0) + (tx + (1 - t)y)^\alpha \eta(f(1), f(0)) \\ &= t^\alpha f(0) + (1 - t)^\alpha f(0) + t^\alpha x^\alpha \eta(f(1), f(0)) + (1 - t)^\alpha y^\alpha \eta(f(1), f(0)) \\ &= t^\alpha f(0) + x^\alpha \eta(f(1), f(0)) + (1 - t)^\alpha (f(0) + y^\alpha \eta(f(1), f(0)) \\ &= t^\alpha f(x) + (1 - t)^\alpha f(y). \end{aligned}$$

This shows that f is generalized η-affine. □

Definición 2.10. *Let* $\mathbb{R}_+^\alpha$ *be the* α*-type set of all nonnegative real numbers and let* $\kappa : \mathbb{R} \times \mathbb{R} \times [0, 1] \to \mathbb{R}_+^\alpha$ *a function with* $0^\alpha \leq t^\alpha \kappa(x, y, t) \leq 1^\alpha$ *for each pair* $x, y \in \mathbb{R}$ *and* $t \in [0, 1]$. *A function* $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}^\alpha$ *is called generalized* η_κ*-convex, if*

$$f(tx + (1 - t)y) \leq f(y) + t^\alpha \kappa(x, y, t)\eta(f(x), f(y))$$

for all $x, y \in \mathbb{R}$ *and* $t \in [0, 1]$.

Teorema 2.11. *For a function* $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}^\alpha$ *the following statements are equivalent:*

(1) f *is generalized* η_κ*-convex for some function* κ.

(2) *f is generalized η-quasiconvex.*

Demostración. (1) $\Rightarrow$ (2) For $x, y \in I$ and $t \in [0, 1]$,

$$f(tx + (1-t)y) \leq f(y) + t^{\alpha}\kappa(x, y, t)\eta(f(x), f(y)) \leq \text{máx}\{f(y), f(y) + \eta(f(x), f(y))\}.$$

(2) $\Rightarrow$ (1) For $x, y \in I$ and $t \in [0, 1]$ we define

$$\kappa(x, y, t) = \begin{cases} (1/t)^{\alpha}, & \text{if } t \in (0, 1] \text{ and } f(y) \leq f(y) + \eta(f(x), f(y)); \\ 0^{\alpha}, & \text{if } t = 0 \text{ or } f(y) > f(y) + \eta(f(x), f(y)). \end{cases}$$

Observe that $0^{\alpha} \leq t^{\alpha}\kappa(x, y, t) \leq 1^{\alpha}$. For a such function κ we have

$$\begin{aligned} f(tx + (1-t)y) &\leq \text{máx}\{f(y), f(y) + \eta(f(x), f(y))\}. \\ &= t^{\alpha}\kappa(x, y, t)(f(y) + \eta(f(x), f(y)) + (1^{\alpha} - t^{\alpha}\kappa(x, y, t))f(y) \\ &= f(y) + t^{\alpha}\kappa(x, y, t)\eta(f(x), f(y)), \end{aligned}$$

which shows that f is generalized η_{κ}-convex. □

Teorema 2.12. *Let $f \in \eta\text{-}GC_{\alpha}(I)$ and let f with local fractional derivate of orden α. If $x \in I$ is the minimum of f, then*

$$f^{\alpha}(x)\frac{(y-x)^{\alpha}}{\Gamma(1+\alpha)} \geq 0^{\alpha} \implies \eta(f(y), f(x)) \geq 0^{\alpha}$$

.

Demostración. Let $x \in I$ the minimum of the function $f \in \eta\text{-}GC_{\alpha}(I)$, then $f(x) \leq f(y)$ for any $y \in I$. Since $I = [a, b]$ e is a convex subset of $\mathbb{R}$, then for each $t \in [0, 1]$ we have $y_t = x + t(y - x) \in I$. Then, $0^{\alpha} \leq f(x + t(y - x)) - f(x)$ and dividing above inequality by t^{α} we obtain that

$$0^{\alpha} \leq \frac{f(x + t(y-x)) - f(x)}{t^{\alpha}}. \tag{1}$$

Setting $h = t(y - x)$, we have that $t = \frac{h}{y-x}$ and hence,

$$0^{\alpha} \leq \frac{f(x+h) - f(x)}{h^{\alpha}}(y-x)^{\alpha} = \frac{\Gamma(1+\alpha)[f(x+h) - f(x)]}{h^{\alpha}} \cdot \frac{(y-x)^{\alpha}}{\Gamma(1+\alpha)}. \tag{2}$$

If $h \to 0$, it follows that

$$0^{\alpha} \leq f^{\alpha}(x)\frac{(y-x)^{\alpha}}{\Gamma(1+\alpha)} \tag{3}$$

Since $f \in \eta\text{-}GC_{\alpha}(I)$, then

$$f(x + t(y-x)) = f(ty + (1-t)x) \leq f(x) + t^{\alpha}\eta(f(y), f(x)),$$

which implies that

$$\frac{f(x + t(y-x)) - f(x)}{t^{\alpha}} \leq \eta(f(y), f(x)).$$

Taking limit on both sides as $t \to 0$, we obtain that

$$f^{\alpha}(x)\frac{(y-x)^{\alpha}}{\Gamma(1+\alpha)} \leq \eta(f(y), f(x)).$$

By (3) we conclude that

$$0^{\alpha} \leq \eta(f(y), f(x)),$$

with this the proof is complete. □

The following result is a new refinement of the Hermite-Hadamard type inequality via generalized η-convex functions.

Teorema 2.13. *Let $f \in \eta\text{-}GC_\alpha([a,b])$. If $\eta(.,.)$ is bounded from above by M_η^α on $f([a,b]) \times f([a,b])$, then*

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) - \left(\frac{M_\eta}{2}\right)^\alpha &\leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} \, {}_aI_b^{(\alpha)} f(x) \\
&\leq \frac{f(a)+f(b)}{2^\alpha} + \frac{\eta(f(a),f(b)) + \eta(f(b),f(a))}{2^\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \\
&\leq \frac{f(a)+f(b)}{2^\alpha} + M_\eta^\alpha \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}.
\end{aligned}
$$

Demostración. Since $F \in \eta\text{-}GC_\alpha([a,b])$, we have

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &= f\left(\frac{a+b}{4} - \frac{t(b-a)}{4} + \frac{a+b}{4} + \frac{t(b-a)}{4}\right) \\
&= f\left(\frac{1}{2}\left(\frac{a+b-t(b-a)}{2}\right) + \frac{1}{2}\left(\frac{a+b+t(b-a)}{2}\right)\right) \\
&\leq f\left(\frac{a+b-t(b-a)}{2}\right) + \left(\frac{1}{2}\right)^\alpha \eta\left(f\left(\frac{a+b+t(b-a)}{2}\right), f\left(\frac{a+b-t(b-a)}{2}\right)\right) \\
&\leq f\left(\frac{a+b-t(b-a)}{2}\right) + \left(\frac{M_\eta}{2}\right)^\alpha,
\end{aligned}
$$

which implies that

$$
f\left(\frac{a+b-t(b-a)}{2}\right) \geq f\left(\frac{a+b}{2}\right) - \left(\frac{M_\eta}{2}\right)^\alpha,
$$

and similarly

$$
f\left(\frac{a+b+t(b-a)}{2}\right) \geq f\left(\frac{a+b}{2}\right) - \left(\frac{M_\eta}{2}\right)^\alpha.
$$

Now, using the change of variable technique for local fractional integrals of order α, we obtain that

$$
\begin{aligned}
\frac{2^\alpha}{(b-a)^\alpha} \int_a^b f(x)\,(dx)^\alpha &= \frac{2^\alpha}{(b-a)^\alpha} \left[\int_a^{(a+b)/2} f(x)\,(dx)^\alpha + \int_{(a+b)/2}^b f(x)\,(dx)^\alpha\right] \\
&= \int_0^1 f\left(\frac{a+b-t(b-a)}{2}\right)(dt)^\alpha + \int_0^1 f\left(\frac{a+b+t(b-a)}{2}\right)(dt)^\alpha \\
&= \int_0^1 \left[f\left(\frac{a+b-t(b-a)}{2}\right) + f\left(\frac{a+b+t(b-a)}{2}\right)\right](dt)^\alpha \\
&\geq \int_0^1 \left[2^\alpha f\left(\frac{a+b}{2}\right) - M_\eta^\alpha\right](dt)^\alpha = 2^\alpha \int_0^1 \left[f\left(\frac{a+b}{2}\right) - \left(\frac{M_\eta}{2}\right)^\alpha\right](dt)^\alpha \\
&= 2^\alpha \left[f\left(\frac{a+b}{2}\right) - \left(\frac{M_\eta}{2}\right)^\alpha\right].
\end{aligned}
$$

Therefore,

$$\frac{1}{(b-a)^{\alpha}}\int_a^b f(x)\,(dx)^{\alpha} \quad\geq\quad f\left(\frac{a+b}{2}\right)-\left(\frac{M_{\eta}}{2}\right)^{\alpha}$$

and consequently,

$$\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{(\alpha)}f(x) \quad\geq\quad f\left(\frac{a+b}{2}\right)-\left(\frac{M_{\eta}}{2}\right)^{\alpha}.$$

On the other hand, as $f\in\eta\text{-}GC_{\alpha}([a,b])$, we have

$$f(ta+(1-t)b)\leq f(b)+t^{\alpha}\eta(f(a),f(b)).$$

Now, applying local fractional integration of order α with respect to t on $[0,1]$:

$$\int_0^1 f(ta+(1-t)b)\,(dt)^{\alpha} \quad\leq\quad \int_0^1 \left[f(b)+t^{\alpha}\eta(f(a),f(b))\right]\,(dt)^{\alpha},$$

which implies that

$$\begin{aligned}\frac{1}{(b-a)^{\alpha}}\int_a^b f(x)\,(dx)^{\alpha} &\leq f(b)+\eta(f(a),f(b))\int_0^1 t^{\alpha}\,(dt)^{\alpha}\\ &= f(b)+\eta(f(a),f(b))\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}=A.\end{aligned}$$

Also

$$\frac{1}{(b-a)^{\alpha}}\int_a^b f(x)\,(dx)^{\alpha} \quad\leq\quad f(c)+\eta(f(b),f(a))\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}=B.$$

Thus, we obtain

$$\begin{aligned}\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{(\alpha)}f(x) &\leq \text{mín}\{A,B\}\\ &\leq \frac{f(a)+f(b)}{2^{\alpha}}+\frac{\eta(f(a),f(b))+\eta(f(b),f(a))}{2^{\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\\ &\leq \frac{f(a)+f(b)}{2^{\alpha}}+M_{\eta}^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}.\end{aligned}$$

This completes the proof. □

Observación 2.14. *It is important to note that if $\eta(x^{\alpha},y^{\alpha})=x^{\alpha}-y^{\alpha}$ then Theorem 2.13 becomes a result for the generalized convex functions introduced in [9].*

Definición 2.15. *A function $f:[a,b]\to\mathbb{R}^{\alpha}$ is said to be symmetric with respect to $\frac{a+b}{2}\in[a,b]$, if*

$$f(x)=f(a+b-x)$$

for all $x\in[a,b]$.

The following result is a Fejér type inequality for generalized η-convex functions.

Teorema 2.16. *Let $f \in \eta\text{-}GC_\alpha([a,b])$. If $\eta(.,.)$ is bounded from above on $f([a,b]) \times f([a,b])$. Moreover, suppose that $w : [a,b] \to \mathbb{R}_+^\alpha$ is symmetric with respect to $\frac{a+b}{2}$ and $w \in I_x^{(\alpha)}[a,b]$, then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) {}_aI_b^{(\alpha)}w(x) - L_\eta(a,b) &\leq {}_aI_b^{(\alpha)}f(x)w(x) \\ &\leq \frac{f(a)+f(b)}{2^\alpha} {}_aI_b^{(\alpha)}w(x) + R_\eta(a,b), \end{aligned}$$

where

$$L_\eta(a,b) := \frac{1}{2^\alpha} {}_aI_b^{(\alpha)}\eta(f(a+b-x), f(x))w(x),$$

and

$$R_\eta(a,b) := \frac{\eta(f(a),f(b)) + \eta(f(b),f(a))}{2^\alpha(b-a)^\alpha} {}_aI_b^{(\alpha)}(b-x)^\alpha w(x),$$

respectively.

Demostración. Since $f \in \eta\text{-}GC_\alpha([a,b])$, we have

$$f\left(\frac{a+b}{2}\right) \leq f((1-t)a+tb) + \frac{1}{2^\alpha}\eta(f((1-t)b+ta), f((1-t)a+tb)).$$

Using the fact that $w \in I_x^{(\alpha)}[a,b]$ and the fact that w is symmetric with respect to $\frac{a+b}{2}$, we obtain that

$$\begin{aligned} f\left(\frac{a+b}{2}\right) {}_aI_b^{(\alpha)}w(x) &= f\left(\frac{a+b}{2}\right)(b-a)^\alpha {}_0I_1^{(\alpha)}w((1-t)a+tb) \\ &\leq (b-a)^\alpha {}_0I_1^{(\alpha)}f((1-t)a+tb)w((1-t)a+tb)) \\ &\quad + \frac{(b-a)^\alpha}{2^\alpha} {}_0I_1^{(\alpha)}\eta(f((1-t)b+ta), f((1-t)a+tb))w((1-t)a+tb) \\ &= {}_aI_b^{(\alpha)}f(x)w(x) + \frac{1}{2^\alpha} {}_aI_b^{(\alpha)}\eta(f(a+b-x), f(x))w(x). \end{aligned} \tag{4}$$

This shows the inequality of the left side of the theorem. Now, again using the fact $w \in I_x^{(\alpha)}[a,b]$ and the fact that w is symmetric with respect to $\frac{a+b}{2}$, we have that

$$\begin{aligned} {}_aI_b^{(\alpha)}f(x)w(x) &\leq (b-a)^\alpha {}_0I_1^{(\alpha)}[f(b) + t^\alpha\eta(f(a),f(b))]w(ta+(1-t)b) \\ &\leq (b-a)^\alpha {}_0I_1^{(\alpha)}f(b)w(ta+(1-t)b) \\ &\quad +(b-a)^\alpha\eta(f(a),f(b)) {}_0I_1^{(\alpha)}t^\alpha w(ta+(1-t)b). \end{aligned} \tag{5}$$

Similarly,

$$\begin{aligned} {}_aI_b^{(\alpha)}f(x)w(x) &\leq (b-a)^\alpha {}_0I_1^{(\alpha)}f(a)w(ta+(1-t)b) \\ &\quad +(b-a)^\alpha\eta(f(b),f(a)) {}_0I_1^{(\alpha)}t^\alpha w(ta+(1-t)b). \end{aligned} \tag{5}$$

Then, adding (4) y (5), we obtain that

$$\begin{aligned} 2^\alpha {}_aI_b^{(\alpha)}f(x)w(x) &\leq (b-a)^\alpha[f(a)+f(b)] {}_aI_b^{(\alpha)}w(ta+(1-t)b) \\ &\quad +(b-a)^\alpha[\eta(f(a),f(b)) + \eta(f(b),f(a))] {}_0I_1^{(\alpha)}t^{(\alpha)}w(ta+(1-t)b). \end{aligned}$$

Applying the change of variable technique for local fractional integration of order α, we conclude that

$$ {}_aI_b^{(\alpha)} f(x)w(x) \quad \leq \quad \frac{f(a)+f(b)}{2^\alpha}\ {}_aI_b^{(\alpha)} w(x) + \frac{\eta(f(a),f(b))+\eta(f(b),f(a))}{2^\alpha(b-a)^\alpha}\ {}_aI_b^{(\alpha)} (b-x)^\alpha w(x), $$

which completes the proof. □

Observación 2.17. *If $w(x) = 1^\alpha$ then Theorem 2.16 reduces to Theorem 2.13. Observe that if $\eta(x^\alpha, y^\alpha) = x^\alpha - y^\alpha$, then Theorem 2.16 is a Hermite-Hadamard-Fejér type inequality for generalized convex functions.*

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