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A note on embedding subgroups of products topological groups

Una nota sobre encajes de subgrupos de productos de grupos topológicos

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Abstract

In this paper, we study ω-narrow and ω-balanced topological groups and prove they may be embedded as subgroups of products of second countable (resp, first countable) topological groups. We also prove that this kind of groups are closed with respect to the most common operations, such as the taking of subgroups, arbitrary products and under continuos homomorphic images. We finally prove that the class of ω-balanced topological groups is wider than the class of ω-narrow topological groups.

Keywords: Topological group, topological group ω-narrow, topological group ω-balanced and topological isomorphism.

Resumen

En este artículo se estudian los grupos topológicos ω-estrechos y ω-balanceados y se demuestra que se pueden encajar como subgrupos de productos de grupos topológicos segundo numerable o primero numerable respectivamente. Se prueba que estas clases de grupos son cerradas bajo las operaciones mas frecuentes en grupos topológicos, son cerradas bajo subgrupos, bajo productos arbitrarios y se conservan atravéz de homomorfismos continuos. Se muestra también que la clase de grupos topológicos ω-balanceados es más amplia que la clase de grupos topológicos ω-estrechos.

Palabras claves: Grupos topológicos, grupos topológicos ω-estrechos, grupos topológicos ω-balacenados e isomorfismos topológicos.

1. Introducción

Tychonoff spaces are spaces that can be embedded in a topological product of the form $I^{\mathfrak{m}}$, where $\mathfrak{m}$ is a sufficiently large cardinal and $I = [0, 1]$. We want to have a similar classification option in topological groups. In this case, since we have additional structure (the algebraic) we can use neither I nor $\mathbb{R}$. We then turn to second numerable groups or first numerable groups. By the theorem of it Birkoff-Kakutani, a topological group is first countable if, and only if, it is metrizable.

The purpose of this paper is to determine under what conditions a topological group can be embedded as a subgroup of a product of second countable or first countable topological groups.

2. Preliminary

The terminology of [3], [6], [11] and [12], is used throughout.

A *semigroup* is a non-void set S together with a mapping $(x, y) \rightarrow x \cdot y$ of $S \times S \rightarrow S$ such that $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in S$.

An element e of a semigroup S is called an *identity* for S if $ex = x = xe$ for every $x \in S$. A semigroup with identity is called *monoide*. An element x of a monoid M is said to be *invertible* if there exists an element y of M such that que $x \cdot y = e = y \cdot x$. If every element x of a monoid M is invertible, then M is called *group*.

Let S be a semigroup. For a fixed element $a \in S$,the mappings $x \mapsto ax$ and $x \mapsto xa$ of S to itself are called the *left* and *right actions* of a on S, and denoted by ϱ_a y λ_a respectively.

If G is a group, the mapping $x \mapsto x^{-1}$ of G onto itself is called *inversion*.

- A *right topological semigroup* consists of a semigroup S and a topology τ on S such that for all $a \in S$, the right action ϱ_a of a on S is a continuous mapping of the space S to itself.
- A *left topological semigroup* consists of a semigroup S and a topology τ on S such that for all $a \in S$, the left action λ_a of a on S is a continuous mapping of the space S to itself.
- A *semigroup topological* consists of a semigroup S and a topology τ on S such that the multiplication in S, as a mapping of $S \times S$ to S, is continuos when $S \times S$ is endowed with the product topology.
- A *right topological monoid* is a right topological semigroup with identity. Similarly, a *topological monoid* is a topological semigroups with identity, and a *semitopological monoid* is a semitopological semigroup with identity.
- A *left*(*right*) *topological group* is a left (right) topological semigroup whose underlying semigroup is a group, and a *semitopological group* is a left topological group which is also a right topological group.
- A topological group G is a topological space that is also a group such that the group operations of product:
  $$G \times G \rightarrow G\colon (x, y) \mapsto xy$$
  and taking inverses:
  $$G \rightarrow G : x \mapsto x^{-1}$$
  are continuous.

2.1. Cardinals invariants

Cardinal functions are widely used in topology as a tool for describing various topological properties. Perhaps the simplest cardinal invariants of a topological space X are its cardinality and the cardinality of its topology, denoted respectively by $|X|$ and $o(X)$.

Next we will give a list of the most known cardinal functions.

- *Weight*: $w(G) = \text{mín}\{|\mathcal{B}|: \mathcal{B} \text{ is base of } G\} + \aleph_0$.
- *Density*: $d(G) = \text{mín}\{|D|: D \text{ dense subset of } G\} + \aleph_0$.
- *Cellularity or Suslin number*: $c(G) = \sup\{|\mathcal{V}|: \mathcal{V} \text{ is a family of mutually disjoint non-empty open subsets of } G\} + \aleph_0$.
- *Network weight* : $nw(G) = \text{mín}\{|\mathcal{U}|: \mathcal{U} \text{ is network for } G\} + \aleph_0$.
- *Character*: $\chi(G) = \sup\{\chi(p, G)^1 : p \in G\} + \aleph_0$.
- *π-Character*: $\pi\chi(G) = \sup\{\pi\chi(p, G)^2 : p \in G\} + \aleph_0$.
- *Pseudocharacter*: $\psi(G) = \sup\{\psi(p, G)^3 : p \in G\} + \aleph_0$.
- *Tightness*: $t(G) = \sup\{t(p, G)^4 : p \in G\} + \aleph_0$.

3. Main result

In trying to give a characterization of the subgroups of Lindelöf groups similar to the existing subgroups of compact groups, I. Guran [7] introduces the notion of group ω- bounded. In recent years, this name was changed to that of ω-narrow because it already exists a different notion with that name. The class of the ω-narrow groups did not characterize the subgroups of Lindelöf groups, but having very stable properties concerning the main operations between topological groups has taken great relevance in the topological algebras.

In this section, our goal is to study the ω-balanced and ω-narrow topological groups and their most important properties.

3.1. ω-narrow topological group

Definition 3.1. *A topological group G is called ω-narrow if, for every open neighbourhood U of the neutral element e in G, there exists a countable subset* $A \subseteq G$ *such that* $G = A \cdot U$.

Next, we show that the algebraic asymmetry of the previous definition disappears, in the case of topological groups, it is more yet disappears in the case of quasi-topological groups.

Proposition 3.2. *The following conditions are equivalent for a topological group G:*

1) G is ω-narrow;

[1] $\chi(p, G) = \text{mín}\{|\mathcal{V}|: \mathcal{V} \text{ local base for } p\}$;
[2] $\pi\chi(p, G) = \text{mín}\{|\mathcal{V}|: \mathcal{V} \text{ is a } \pi\text{-local base for } p\}$;
[3] $\psi(p, G) = \text{mín}\{|\mathcal{V}|: \mathcal{V} \text{ is a pseudobase for } p\}$;
[4] $t(p, G) = \text{mín}\{|\tau|: p \in \overline{C}, C \subseteq G, \text{exist } F \subseteq S \text{ such that } p \in \overline{F} \text{ y } |F| \leq \tau\}$;

2) *For every open neighbourhood V of the neutral element e in G, there exists a countable set $B \subset G$ such that $G = VB$;*

3) *For every open neighbourhood V of the neutral element e in G, there exists a countable set $C \subset G$ such that $CV = G = VC$.*

Demostración. See [3], proposition 3.4.1. □

Proposition 3.3. *If a topological group H is a continuous homomorphic image of ω-narrow topological group G, then H is also ω-narrow.*

Demostración. Let $\pi\colon G \to H$ a continuous homomorphism of a ω-narrow topological group G on a topological group H. Let V an open neighbourhood of the neutral element of H. The set $U = \pi^{-1}(V)$ is open in G. Therefore, there is a countable subset A of G such that $A \cdot U = G$. We see that the set $\pi(A) = B$ is countable and $B \cdot V = \pi(A \cdot U) = \pi(G) = H$. Therefore H is ω-narrow. □

Proposition 3.4. *The topological product of an arbitrary family of ω-narrow topological group is an ω-narrow.*

Demostración. Let $G = \Pi_{i\in I} G_i$ be a family of topological groups ω-narrow G_i. If U is open neighbourhood of the neutral element e in G, there is an open canonical set V in G such that $e \in V \subseteq U$. Let $i_1, \ldots, i_n \in I$ the coordinates that satisfy equality $V = p_{i_1}^{-1} p_{i_1}(V) \cap \cdots \cap p_{i_n}^{-1} p_{i_n}(V)$, $(*)$, where p_i is the projection of G on the factor G_i, $i \in I$. Note that $V_k = P_{i_k}(V)$ is open neighbourhood of the neutral element e in G_{i_k} for all $k \leq n$, let's choose a countable subset C_k de G_{i_k} so that $C_k \cdot V_k = G_{i_k}$. We define the set C of G through$C = \Pi_{i\in I} A_i$, donde $A_i = C_k$ si $i = i_k$ for some $k \leq n$, y $A_i = \{e_{G_i}\}$ otherwise. It is clear that $|C| = |C_1 \times \cdots \times C_n| \leq \aleph_0$. Of $(*)$ it follows that $C \cdot V = G$, and in consequence $C \cdot U = G$. Therefore G is ω-narrow. □

Theorem 3.5. *Every subgroup H of an G ω-narrow topological group is ω-narrow.*

Demostración. Let G an ω-narrow topological grouop and let H an subgroup of G. Be an open neighbourhood U of the identity e in H. Let's choose an asymmetric open neighborhood V of e in G such that $V^2 \cap H \subset U$. Since G is ω-narrow, there is a countable subset B of G such that $G = B \cdot V$. Let $C = \{c \in B : cV \cap H \neq \emptyset\}$. Then $|C| \leq |B| \leq \omega$, It is clearly that $H \subset CV$. For all $c \in C$ fix, choose $a_c \in cV \cap H$ and let $A = \{a_c : c \in C\}$. Since C is countable the subset A of H is countable. We affirm that $H = A \cdot U$. As H is a subgroup of G and $V^2 \cap H \subset U \subset H$, we have to $(AV^2) \cap H \subset A \cdot U \subset H$. Let $H \subset A \cdot V^2$. Clearly, $A \subset H \subset CV$. Given that V It is symmetric, it follows that $C \subset A \cdot V$, It is symmetric, it follows that $H \subset CV \subset A \cdot V^2$. Therefore $H \subset A \cdot U$. So $H = A \cdot U$ which means that H is ω-narrow. □

The proposition 3.3, 3.4 and the theorem 3.5, shows us that the class of topological groups ω-narrow is stable under the most frequent operations in topological groups.

Remark 3.6. *To give an example of a topological group that is not ω-narrow is very simple, just take any discrete topological group that is not countable.*

The observation before can be seen more generally in the following proposition.

Proposition 3.7. *Every first-countable ω-narrow topological group has a countable base.*

Demostración. Let $\eta = \{U_n : n \in \omega\}$ be a countable base at the identity e of an ω-narrow topological group G ω-estrecho. For all $n \in \omega$ choose a countable set $C_n \subset G$ such that $G = C_n \cdot U_n$. Then the family $\beta = \{xU_n : x \in C_n, n \in \omega\}$ is countable, and we claim that β is a base for the group G. Indeed, let O be a neighbourhood of a point $a \in G$. One can find $k, l \in \omega$ such that $aU_k \subset O$ and $U_l^{-1}U_l \subset U_k$. There exists $x \in C_l$ such that $a \in xU_l$, whence $x \in aU_l^{-1}$. We have $xU_l \subset (aU_l^{-1})U_l = a(U_l^{-1}U_l) \subset aU_k \subset O$, that is, xU_l is an open neighbourhood of a and $xU_l \subset O$. It remains to note that $xU_l \in \beta$. □

Proposition 3.8. *Every Lindelöf topological group is ω-narrow.*

Demostración. Let G an Lindelöf topological group and U an open neighbourhood e of G. The family $\mathcal{U} = \{xU : x \in G\}$ it is an open cover of G. As G is Lindelöf, there is a countable subfamily $\mathcal{U}'$ of $\mathcal{U}$ that covers G. For definition of $\mathcal{U}$, the above means that $\mathcal{U}' = \{xU : x \in C\}$ covers to G, for a countable subset C of G, that is to say, $G = C \cdot U$. □

Theorem 3.9. *If the cellularity of a topological group G is countable, then G is ω-narrow.*

Demostración. See [3], Theorem 3.4.7. □

Remark 3.10. *Since a separable space has countable cellularity, the previous theorem implies that every separable topological group is ω-narrow topological group. And therefore, the following corollary is established.*

Corollary 3.11. *Every separable topological group is ω-narrow.*

Remark 3.12. *As we have seen the ω -narrowness is inherited to subgroups. On the contrary, the fact that a G group contains a ω-narrow subgroup does not imply that G is. However, this implication is valid if H is a dense subgroup in G.*

Proposition 3.13. *If a topological group G contains a dense subgroup H such that H is ω-narrow, then G is also ω-narrow.*

Demostración. Let U be any open neighbourhood of e in G. There exists a symmetric open neighbourhood V of e in G such that $V^2 \subset U$. Since H is an ω-narrow topological subgroup, we can find a countable subset $C \subseteq H$ such that $H \subseteq C \cdot V$; in particular $C \cdot V$ is dense in G. We claim that $G = C \cdot U$. Let $x \in G$ arbitrary. The set xV it's a neighborhood of x, so that $(xV) \cap (C\dot{V}) \neq \emptyset$. Then there are $v_1, v_2 \in V$ y $c \in C$ such that $xv_1 = cv_2$ and therefore $x = cv_2v_1^{-1} = cv_2v_1 \in C \cdot V \cdot V = C \cdot V^2 \subseteq C \cdot U$. So $G = C \cdot U$, and therefore, G is ω-narrow. □

3.2. ω-balanced topological groups

In 1936, G. Birkhoff and S. Kakutani ([4] and [8]) gave, independently, necessary conditions and enough for a topological group to be metrizable. From the reference [3], we will quote the following theorem, which gives us a characterization for the metrizable topological groups.

Theorem 3.14 (G. Birkhoff, S. Kakutani). *A topological group G is metrizable if and only if it is first-countable.*

Demostración. See [3], Theorem 3.3.12. □

To apply the previous theorem to our paper, we introduce some definitions and propositions.

Definition 3.15. *Let us say that the invariance number* $Inv(G)$[5] *of a topological group G is countable if for each open neighbourhood U of the neutral element e in G, there exists a countable family* γ *of open neighbourhoods of e such that for each* $x \in G$*, there exists* $V \in \gamma$ *satisfying* $xVx^{-1} \subseteq U$*. Any such family* γ *will be called subordinated to U. Topological groups G such that* $Inv(G) \leq \omega$ *are also called* ω*-balanced. Clearly, that every subgroup of an* ω*-balanced group is also* ω*-balanced.*

Remark 3.16. *It is clear that all subsets of Abelian groups are invariant. Therefore all Abelian groups are balanced.*

The following result gives an alternative characterization for balanced groups.

Lemma 3.17. *A topological group G is balanced*[6] *if and only if for every neighborhood U of e, there is a neighborhood V of e such that* $xVx^{-1} \subset U$ *for every* $x \in G$*.*

Demostración. We will only show sufficiency since the need is clear. Let U be any neighborhood of the identity element e of G. Let's take an open neighborhood V of e such that $xVx^{-1} \subset U$ for all $x \in G$. Then set $O = \bigcup_{x \in G} xVx^{-1} \subset U$ is an open in G. We just need to see that the set O is invariant. Indeed, let's take an arbitrary element $y \in G$. Then:

$$yOy^{-1} = y\Big(\bigcup_{x\in G} xVx^{-1}\Big)y^{-1} = \bigcup_{x\in G} yxVx^{-1}y^{-1} = \bigcup_{z\in G} zVz^{-1} = O.$$

The above equals show that the V set is invariant. Therefore the G group has a $\mathcal{B}$ base consisting of invariant open sets. □

The following proposition shows us that the class of ω-balanced is broader than the class of the ω-narrow groups.

Proposition 3.18. *If G is an* ω*-narrow topological group, then the invariance number of G is countable, that is, G is* ω*-balanced.*

Demostración. Let U be an open neighbourhood of the neutral element e in G. There exists a symmetric open neighbourhood V of e such that $V^3 \subset U$. Since G is ω-narrow, we can find a countable subset $A \subseteq G$ such that $G = V \cdot A$. Then for each $a \in A$, there exists an open neighbourhood W_a of the neutral element e such that $aW_aa^{-1} \subset V$. We claim that $\gamma = \{W_a : a \in A\}$ is the family we are looking for. Indeed, γ is a countable family of open neighbourhoods of e. Now, let x be any element of G. Entonces $x \in Va$, for some $a \in A$, and therefore, $xW_ax^{-1} \subset VaW_aa^{-1}V^{-1} \subset V \cdot V \cdot V^{-1} \subset V^3 \subset U$, that is, γ is subordinated to U. □

The converse to the previous statement is not true. Indeed, every discrete group is obviously ω-balanced, while a discrete group is ω-narrow if and only if it is countable.

Proposition 3.19. *The invariance number of an arbitrary first-countable topological group G is countable.*

Demostración. Let $\{V_n : n \in \omega\}$ be a countable base of the space G at the neutral element e of the group G. Take any open neighbourhood U of e. Then Ux and xU are an open neighbourhoods of x. Since the left and right translations by x are continuous and $xe \in Ux$, $ex \in xU$, there exists $n \in \omega$ such that $xV_n \subset Ux$ and $V_nx \subset xU$. It follows that $xV_nx^{-1} \subset Uxx^{-1} = U$ and $x^{-1}V_nx \subset x^{-1}xU = U$. Therefore, $Inv(G) \leq \omega$ □

[5] Notation: $Inv(G) \leq \omega$.

[6] Let G an topological group. An subset A of G is called *invariant* if $xAx^{-1} = A$ for all $x \in G$. We say that the topological group G is *balanced*, if you have a local base $\mathcal{B}$ of the neutral element e consisting of invariant sets.

Corollary 3.20. *Every metrizable topological group is ω-balanced.*

Now we will enunciate two lemmas that will be very useful to us.

Lemma 3.21. *Let G be an ω-balanced topological group, and let γ be a countable family of open neighbourhoods of the neutral element e in G. Then there exists a countable family γ^* of open neighbourhoods of e with the following properties:*

1) $\gamma \subset \gamma^*$;
2) the intersection of any finite subfamily of γ^* belongs to γ^*;
3) for each $U \in \gamma^*$, there exists a symmetric $V \in \gamma^*$ such that $V^2 \subset U$;
4) for every $U \in \gamma^*$ and every $a \in G$, there exists $V \in \gamma^*$ such that $aVa^{-1} \subset U$.

Demostración. See [3], Lemma 3.4.13. □

Now we easily obtain the next lemma designed for a direct application.

Lemma 3.22. *Let G be an ω-balanced topological group, and U an open neighbourhood of the neutral element e in G. Then there exists a sequence $\{U_n : n \in \omega\}$ of open neighbourhoods of e such that, for each $n \in \omega$, the following conditions are satisfied:*

a) $U_0 \subset U$;
b) $U_n^{-1} = U_n$;
c) $U_{n+1}^2 \subset U_n$;
d) For each $x \in G$ and each $n \in \omega$, there is $k \in \omega$ such that $xU_kx^{-1} \subset U_n$.

Demostración. See [3], Lemma 3.4.14. □

The following theorem shows us that the nullity of a pseudo-metric continuous left-invariant is a normal and closed subgroup.

Theorem 3.23. *Let G be an ω-balanced topological group. Then, for every open neighbourhood U of the neutral element e in G, there exists a continuous left-invariant pseudometric on G such that the following conditions are satisfied:*

$(p1)$ $\{x \in G : \rho(e, x) < 1\} \subset U$;

$(p2)$ $\{x \in G : \rho(e, x) = 0\}$ *is a closed invariant subgroup of G;*

$(p3)$ *for any $x, y \in G$, $\rho(e, xy) \leq \rho(e, x) + \rho(e, y)$.*

Demostración. By Lemma 3.22, we can find a sequence $\{U_n : n \in \omega\}$ of open neighbourhoods of e in G satisfying conditions $(a) - (d)$ of the lemma. According to Lemmas 3.22 and 3.21, there exists a continuous prenorm N on G such that the next condition is satisfied:

$$\text{(PN4)} \quad \{x \in G : N(x) < 1/2^n\} \subset U_n \subset \{x \in G : N(x) \leq 2/2^n\} \tag{1}$$

Now, for arbitrary x and y in G, put $\rho(x, y) = N(x^{-1}y)$. Then the continuity of N implies that ρ is also continuous. It is also clear from 1 and condition (a) of Lemma 3.22 that $(p1)$ is satisfied.

Claim 1: *ρ is a pseudometric on the set G.*

Indeed, for any x and y in G we have: $\rho(x, y) = N(x^{-1}y) \geq 0$ and $\rho(y, x) = N(y^{-1}x) = N((y^{-1}x)^{-1}) = N(x^{-1}y) = \rho(x, y)$. Also $\rho(x, x) = N(x^{-1}x) = N(e) = 0$. Further, for any x, y, z en G, we have:

$$\begin{aligned}\rho(x, z) &= N(x^{-1}z) = N(x^{-1}yy^{-1}z) \leq N(x^{-1}y) + N(y^{-1}z)\\ &= \rho(x, y) + \rho(y, z)\end{aligned}$$

Hence, ρ satisfies the triangle inequality.

Claim 2: The pseudometric ρ is left-invariant.

Indeed, $\rho(zx, zy) = N((zx)^{-1}zy) = N(x^{-1}z^{-1}zy) = N(x^{-1}y) = \rho(x, y)$, for arbitrary x, y, and z in G. Put $Z = \{x \in G : N(x) = 0\}$. Notice that $\rho(e, x) = N(x)$, for each $x \in G$, since $\rho(e, x) = N(e^{-1}x) = N(x)$. Therefore, we have that $Z = \{x \in G : \rho(e, x) = 0\}$.

Claim 3: $Z = \bigcap_{n\in\omega} U_n$.

This clearly follows from condition (PN4).

Claim 4:Z is a closed invariant subgroup of G.

Since the prenorm N is continuous, the set Z is closed in the space G. The fact that Z is a subgroup of G follows from Proposition 3.3.4 in [3]. It remains to show that the subgroup Z of G is invariant. Take any $x \in G$. We have to check that $xZx^{-1} \subset Z$. In view of Claim 3, it suffices to show that $xZx^{-1} \subset U_n$, for each $n \in \omega$. Fix $n \in \omega$. From condition (d) of Lemma 3.22 it follows that there exists $k \in \omega$ such that $xU_kx^{-1} \subset U_n$. Since $Z \subset U_k$, we conclude that $xZx^{-1} \subset U_n$, that is, Z invariant.

It remains to notice that condition (p3) is obviously satisfied, since N is a prenorm and $\rho(e, x) = N(x)$. □

The following theorem suggests the idea of embedding a ω -balanced group as a product of metrizable groups.

Theorem 3.24. *Let G an group ω-balanced, then for each open neighbourhood U of the neutral element e in G, there exists a continuous homomorphism π of G onto a metrizable group H such that* $\pi^{-1}(V) \subset U$*, or some open neighbourhood V of the neutral element* e^* *in H.*

The above theorem has an important corollary:

Corollary 3.25. *Let G be an ω-narrow group. Then for every neighbourhood U of the identity in G, there exists a continuous homomorphism π of G onto a second-countable topological group H such that* $\pi^{-1}(V) \subset U$*, for some open neighbourhood V of the identity in H.*

Demostración. By Theorem 3.24, one can find a continuous homomorphism π of G a metrizable topological group H and an open neighbourhood V of the identity in H such that $\pi^{-1}(V) \subset U$. From Proposition 3.3, it follows that the group H is ω-narrow. So 3.7 implies that H is second-countable. □

Definition 3.26. *A topological group G is called range-metrizable for every open neighbourhood U of the neutral element e of G, there exists a continuous homomorphism πonto a metrizable group H such that* $\pi^{-1}(V) \subset U$*, for some open neighbourhood V of the neutral element of H.*

Definition 3.27. *Let* $\mathcal{P}$ *be any class of topological groups, and let G be any topological group. Let us say that G is range* $-\mathcal{P}$ *if for every open neighbourhood U of the neutral element e of G, ethere exists a continuous homomorphism π of G to a group* $H \in \mathcal{P}$ *such that* $\pi^{-1}(V) \subset U$*, for some open neighbourhood V of the neutral element* e^* *of H.*

Remark 3.28. *It follows immediately from the definition that every subgroup of a range* $-\mathcal{P}$ *group is also range* $-\mathcal{P}$*.*

The next fact follows from the definition of the product topology.

Proposition 3.29. *Let $\mathcal{P}$ be any class of topological groups closed under finite products, and let H be the topological product of a family $\{H_a : a \in A\}$ of groups in the class $\mathcal{P}$. Then every subgroup of H is range$-\mathcal{P}$.*

Theorem 3.30. *Let $\mathcal{P}$ be a class of topological groups, τ an infinite cardinal number, and G a topological group, which is range$-\mathcal{P}$ and has a base $\mathcal{B}$ of open neighbourhoods of the neutral element such that $|\mathcal{B}| \leq \tau$. Then G is topologically isomorphic to a subgroup of the product of a family $\{H_a : a \in A\}$ of groups such that $H_a \in \mathcal{P}$, for each $a \in A$ and $|A| \leq \tau$.*

Demostración. We fix a base $\mathcal{B}$ dof open neighbourhoods of the neutral element in G such that $|\mathcal{B}| \leq \tau$. As well as G eis *range-$\mathcal{P}$*, for all $U \in \mathcal{B}$ there is a continuous homomorphism φ_U of G to a group $H_U \in \mathcal{P}$ such that $(\varphi_U)^{-1}(V) \subset U$, for some open neighbourhood V of the neutral element in H. Let's define φ as the diagonal product of the family $\{\varphi_U : U \in \mathcal{B}\}$. We affirm that φ is a topological isomorphism of G nto a topological subgroup of the topological product of the family $\{H_U : U \in \mathcal{B}\}$. Indeed, if we prove that the φ function is injective and that it also separates closed points, we will have proven that φ is a topological isomorphism. Let's see that $varphi$ is injective or, equivalently, that $\ker\varphi$ consists only of the element e. Let $g \in G$ with $g \neq e$ choose $U \in \mathcal{B}$ such that $g \notin U$. Let p_U the product projection of the family $\{H_U : U \in \mathcal{B}\}$ to the factor H_U. Consider the neighborhood $W = p_U^{-1}(V)$ of identity in the family product $\{H_U : U \in \mathcal{B}\}$. Of equality $\varphi_U = p_U \circ \varphi$ it has to$\varphi^{-1}(W) = \varphi_U^{-1}(V) \subseteq U$. Note that $\varphi(g) \notin W$, i.e., $\varphi(g)$ it is different from the identity in the family product $\{H_U : U \in \mathcal{B}\}$, is that $\ker\varphi$ It consists only of identity. On the other hand, if we take the family $\mathcal{F} = \{\varphi_U : U \in \mathcal{B}\}$ and we prove that it separates closed points, the demonstration is complete. Let e of the neutral element of G, C a closed such that $e \notin C$ y $U \in \mathcal{B}$. Then $e \in U \subset G \setminus C$. Now is $\varphi_U : G \to H_U$ and $V \in H$ are such that $\varphi_U^{-1}(V) \subset U \subset G \setminus C$, then $\varphi_U(e) \in V$. To prove that $V \cap \varphi_U(C) = \emptyset$. Indeed be $v \in V$. Then $\varphi_U^{-1}(v) \subset \varphi^{-1}(V) \subset U \subset G \setminus C$ and so $v \in V \subset \varphi(G \setminus C) = \varphi(G) \setminus \varphi(C)$, therefore, $v \notin \overline{\varphi(C)}$. So, lthe family $\mathcal{F}$ closed separate closed points which implies that φ It is a homeomorphic embedded. □

The following two theorems allow us to characterize the topological groups that can be embedded as subgroups of products of first numerable and second numerable groups respectively.

Theorem 3.31 (G. I. Katz.[9] 1953). *every topological group G, the following three conditions are equivalent:*

1) *$Inv(G) \leq \omega$;*

2) *G is range-metrizable;*

3) *G is topologically isomorphic to a subgroup of a topological product of metrizable groups.*

Demostración. It follows from Proposition 3.29 and Theorem 3.30 that 2) and 3) are equivalent. Theorem 3.24 gives the implication 1) and 2) are equivalent. To show that 2) and 1) are equivalent. Take an open neighbourhood U of the neutral element e of G in a range-metrizable group G and consider a continuous homomorphism $p : g \to H$ onto metrizable group H such that $p^{-1}(V) \subset U$ for some open neighbourhood V of e^* in H. Let also $\mathcal{B}$ be a countable base at e^* in H. therefore, the countable family $\{\pi^{-1}(O) : O \in \mathcal{B}\}$ of open neighbourhoods of e in G is subordinated to U, so that $Inv(G) \leq \omega$. □

Theorem 3.32 (I. I. Guran.[7] 1981). *A topological group G is topologically isomorphic to a subgroup of the topological product of some family of second-countable groups if and only if G is ω-narrow.*

Demostración. "⇒"

Let Ω be the class of second-countable topological groups. by Corollary 3.25, every ω-narrow group G is *range*-Ω, it follows from Theorem 3.30 that G is topologically isomorphic to a subgroup of the product of a family of groups from Ω.

"⇐"

By Proposition 3.4) and Theorem 3.5, every subgroup of a topological product of second-countable topological groups is ω-narrow. □

Definition 3.33. *A compact space X is called Dugundji if for every zero-dimensional compact space Z and every continuous mapping $f\colon A \to X$, where A is a closed subset of Z, here exists a continuous mapping $g\colon Z \to X$ extending f.*

The following theorem provides us with a variety of natural examples of topological groups which are not ω-balanced.

Theorem 3.34. *Suppose that X is a zero-dimensional homogeneous compact space such that the group $Homeo(X)$ of all homeomorphisms of X onto itself, with the compact open topology, is ω-balanced. Then X is Dugundji.*

Demostración. See [3], Theorem 10.3.10. □

As a corollary we have the following.

Corollary 3.35. *Suppose that X is a zero-dimensional homogeneous compact space of countable tightness such that the group $Homeo(X)$ of all homeomorphisms of X onto itself, with the compact-open topology, is ω-balanced. Then X is metrizable.*

Demostración. See [3], Theorem 10.3.11. □

The following example meets all the hypotheses of the previous corollary, but nevertheless the conclusion is not fulfilled.

Example 3.36. *Let X the space of two arrows, that is $X = C_0 \cup C_1 \subset \mathbb{R}^2$, where $C_0 = \{(x,0) : 0 < x \leq 1\}$ and $C_1 = \{(x,1) : 0 \leq x < 1\}$, and the topology onto X it is generated by the base consisting of the sets of the form:*

$$\{(x,i) \in X : x_0 - 1/k < x < x_0 \; i = 0,1\} \cup \{(x,0)\},$$

where $0 < x_0 \leq 1$ y $k = 1,2,\ldots$, and the sets of the form:

$$\{(x,i) \in X : x_0 < x < x_0 + 1/k \; i = 0,1\} \cup \{(x,1)\},$$

where $0 \leq x_0 < 1$ y $k = 1,2,\ldots$.

Let $G = Homeo(X)$ the group of all homeomorphisms of X on itself, with the compact-open topology, that is the topology $\tau(X,X) = \mathcal{T}$ generated by the base consisting of all the sets $\bigcap_{i=1}^{k} M(C_i, U_i)$, where C_i it is a compact subset of X and U_i it is an open subset of X for $i = 1,2,\ldots,k$. Then $(G,\mathcal{T})$ it is a topological group which is not ω-balanced.

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