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# Basic and Classic properties in the $\mathfrak{B}_F$-spaces

# Propiedades Básicas y Clásicas en los $\mathfrak{B}_F$-espacios

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## Abstract

$\mathfrak{B}_F$-spaces determine a class between the class of pseudocompact spaces and the class of $k_R$-pseudocompact spaces. We present an alternative proof of the theorem 3.5 enunciated in [3] and describe their main properties.

*Keywords:* $k_R$-space, $\mathfrak{B}_F$-spaces, pseudocompact spaces.

## Resumen

Los espacios $\mathfrak{B}_F$ determinan una clase entre la clase de espacios pseudocompactos y la clase de espacios $k_R$ -pseudocompact. Presentamos una prueba alternativa del teorema 3.5 enunciado en [3] y describimos sus propiedades principales.

*Palabras claves:* $k_R$-espacio, $\mathfrak{B}_F$-espacio, espacio pseudocompacto.

## 1. Introduction

The class of $\mathfrak{B}_F$-spaces lies between the class of pseudocompact spaces and the class of pseudocompact $k_R$-spaces. The difinition of $\mathfrak{B}_F$-spaces 2.2 was introduced by Frolík in [[3], 3.5.1], where he proves that such spaces are productively pseudocompact. The class was later studied by Noble [7], who doesn't give it a name but denotes it by $\mathfrak{B}^*$ ($\mathfrak{B}$ is used for the class of productively pseudocompact spaces by both Frolík and Noble.)

It has several attractive properties like the following:

a) $\mathfrak{B}_F$-spaces are productively pseudocompact;

b) $\mathfrak{B}_F$-spaces are closed under finite products;

c) Every product of $\mathfrak{B}_F$-spaces is pseudocompact;

d) $\mathfrak{B}_F$-spaces are closed under continuous images;

e) Every space containig a dense $\mathfrak{B}_F$-subspace is itself $\mathfrak{B}_F$-spaces

We think all of these facts prove that this is a challenging area in point set topology.

## 2. Preliminary

The terminology of R. Engelking [2] and J. Kelley [6], *General Topology*, is used throughout.
All spaces consider in this paper are *Tychonoff*, i.e., completely regular and *Hausdorff*.

**Definition 2.1.** *A space $X$ is said to be :*

*i) pseudocompact (see Hewitt [4] ) if (and only if ) every real continuous function on $X$ is bounded, or equivalently, if every real continuous bounded function assumes its bounds. A completely regular space $X$ is pseudocompact if and only if every locally finite family of its open subsets is finite, or equivalently, if there exists no locally finite sequence of its non-void open subsets.*

*ii) $k_R$-space(see Noble [7]) when every real-valued function with domain $X$ is continuous if its restriction to each compact subset of $X$ is continuous.*

Recall that a space $X$ is called a $k$-space provided each subset of $X$ which meets every compact set in a relatively closed set is itself closed, and that associated with each space $X$ there is a unique $k$-space $kX$[1] having the same underlying set and the same compact sets as $X$ (see [7]).

The following definition is based on Frolík's condition [[3], 3.5.1] which turns out to be equivalent. $\mathfrak{B}_F$-spaces.

**Definition 2.2.** *A space $X$ is a $\mathfrak{B}_F$-space if for every sequence $U_1, U_2, \dots$ of non-empty open sets, there exists a compact set $K \subseteq X$ such that $K \cap U_n \neq \emptyset$ for infinitely many indices $n$.*

We obtain an equivalent definition if we suppose that the open sets $U_n$ are mutually disjoint. To prove this fact, we need a Lemma.

**Lemma 2.3.** *(see also [7]) Let $U_1, U_2, \dots$ be a point finite sequence of non-empty open sets in a space $X$. Then there exists a sequence $T_1, T_2, \dots$ of mutually disjoint non-empty open sets in $X$ and an increasing sequence $n_0 = 0 < n_1 < n_2 \cdots$ such that $T_i \subseteq \bigcup_{j=n_{i-1}+1}^{n_i} U_j$ for each $i \in \omega$*

[1](The space $kX$ is formed by adjoining to the topology on $X$ all those subsets whose complements meet each compact set in a relatively closed set.) When $X$ is a $T_1$-space, $kX$ is also a $T_1$-space; in fact, the identity map from $kX$ to $X$ is always continuous.

**Proof.** By ([3]), we know every subsequence $U_{n_1}, U_{n_2}, \ldots$ of $U_1, U_2, \ldots$ has an irreducible subcover (which may be finite). If some subsequence $\{U_{n_1}, U_{n_2}, \ldots\}$ $(n_1 < n_2 < \cdots)$ is irreducible, we may select, for each $i \in \omega$, a point $x_i \in U_{n_i} - \bigcup_{j \neq i} U_{n_j}$. Choose an open set $W_1$ such that $x_1 \in W_1 \subseteq W_1^- \subseteq U_{n_1}$. Since $x_2 \notin W_1^-$, there exists an open set $W_2$ such that $x_2 \in W_2 \subseteq W_2^- \subseteq U_{n_2} - W_1^-$. Now, since $x_3 \notin W_1^- \cup W_2^-$, there exists an open set $W_3$ such that $x_3 \in W_3 \subseteq W_3^- \subseteq U_{n_3} - (W_1^- \cup W_2^-)$. Continuing this process, we may construct a sequence $W_1, W_2, \ldots$ of mutually disjoint non-empty open sets such that $W_i \subseteq U_{n_i}$ for each $i \in \omega$ and we are thru in this case Suppose then that no subsequence of $U_1, U_2, \ldots$ is irreducible. Therefore, we may find integers $n_0 = 0 < n_1 < n_2 < \cdots$ such that if $W_i = \bigcup\{U_j : n_{i-1} < j \leq n_i\}$, then $W_1 \supsetneqq W_2 \supsetneqq W_3 \supsetneqq \cdots$. If a subsequence of the $W_i's$ is made of clopen sets, say $W_{k_1}, W_{k_2}, \ldots$ the sequence $\{W_{k_i} - W_{k_{i+1}} : i = 1, 2, \ldots\}$ satisfies our requirements. If only finitely many of the $W_i's$ are clopen, we may remove them and suppose, with no loss of generality, that $W_i \neq W_i^-$ for each $i \in \omega$. If for some strictly increasing sequence $0 < n_1 < n_2 < \cdots$ we have $W_{n_i} - W_{n_{i+1}}^- \neq \emptyset$ for each $i \in \omega$, we define $T_i = W_{n_i} - W_{n_{i+1}}^-$ and the sequence of open sets $T_1, T_2, \ldots$ satisfies our requirements. If for only finitely many indices $i \in \omega$, we have $W_i - W_{i+1}^- \neq \emptyset$, we may remove the corresponding $W_i$ and suppose then that $W_{i+1}$ is dense in $W_i$ for each $i \in \omega$. Take a point $x_1 \in W_1 - W_2$ and let $T_1$ be an open set such that $x_1 \in T_1 \subseteq T_1^- \subseteq W_1$. The set $T_1 \cap W_2$ is then open and infinite. Select two different points $x_2, p_2 \in T_1 \cap W_2$ and let $T_2$ be an open set such that $x_2 \in T_2 \subseteq T_2^- \subseteq T_1 \cap (W_2 - \{p_2\})$. Take now two different points $x_3, p_3 \in T_2 \cap W_3$ and let $T_3$ be an open set such that $x_3 \in T_3 \subseteq T_3^- \subseteq T_2 \cap (W_3 - \{p_3\})$. It is clear now how to continue this process indefinitely. The required sequence is now $\{T_i - T_{i+1}^- : i \in \omega\}$. ■ We prove now the equivalence of the two definitions.

**Proposition 2.4.** *In an arbitrary space X, the following two conditions are equivalent:*

1) *$X$ is a $\mathfrak{B}_F$-space.*
2) *For every open sequence $W_1, W_2, \ldots$ of mutually disjoint non-empty open subsets of X, there exits a compact set $L \subseteq X$ such that $L \cap W_n \neq \emptyset$ for infinitely many indices n.*

**Proof.** We just have to prove that 2) ⇒ 1). Let $U_1, U_2, \ldots$ be a sequence on non-empty open sets of $X$. We may suppose that the sequence $U_1, U_2, \ldots$ is point finite, because otherwise we could take the compact set $K$ as a singleton. By (2.3), there exists a sequence $T_1, T_2, \ldots$ of mutually disjoint non-empty open sets in $X$ and a strictly increasing sequence $n_0 = 0 < n_1 < n_2 < \cdots$ such that $T_i \subseteq \bigcup_{j=n_{i-1}+1}^{n_i} U_j$ for each $i \in \omega$. By property 2), there exists a compact set $K \subseteq X$ such that $K \cap T_i \neq \emptyset$ for infinitely many indices $i \in \omega$. Hence, $K \cap U_j \neq \emptyset$ for infinitely many indices $j \in \omega$ and the proof is complete. ■

**Definition 2.5.** *A subset A of a space X is C-discrete (respect to X) if for each $x \in A$ we may find an open set $U_x$ containing x and such that the family $\{U_x : x \in A\}$ is discrete (respect to X).*

A well known characterization of pseudocompactness is the following:

**Proposition 2.6.** *[see [3]] A space X is pseudocompact if and only if every C-discrete subset of X is finite.*

(2.6) implies immediately:

**Lemma 2.7.** *Every $\mathfrak{B}_F$-space X is pseudocompact.*

**Proof.** Suppose, on the contrary, there exists an infinite discrete sequence $U_1, U_2, \ldots$ of non-empty open subset of $X$. Let $K \subseteq X$ be a compact set such that $K \cap U_n \neq \emptyset$ for every $n \in L$, where $L \subseteq \omega$ and $|L| = \omega$. For each $n \in L$, select a point $x_n \in K \cap U_n$. Then the set $A = \{x_n : n \in L\}$ is an infinite $C$-discrete subset of $K$, contradicting (2.6). ■

We call point $x$ in $X$ is a $k$-point if each open subset of $kX$ which contains $x$ is a neighborhood of $x$. Clearly $X$ is a $k$-space if and only if each point in $X$ is a $k$-point. Recall that a point $x$ in $X$ is called a $P$-point if each $G_\delta$ containing $x$ is a neighborhood of $x$. We call point $x$ in $X$ is a $k_R$-point if each real-valued function on $X$ which is continuous on compact sets is continuous at $x$ (see [7])

We have the following result.

**Proposition 2.8.** *(see [7], theorem 2.2) If $X$ is pseudocompact and each point of $X$ is either a $P$-point or a $k_R$-point, then $X$ is a $\mathfrak{B}_F$-space.*

**Proof.** Suppose $X$ is not $\mathfrak{B}_F$-space, let $\{U_n\}$ be a countable collection of disjoint open sets only finitely many of which meet any single compact set and construct and bounded function $f$ (see [[7], theorem 2.1]). Since $f$ is continuous on compact sets it is continuous at each $k_R$-point of $X$, and $f$ is continuous at $P$-point in $X \setminus \bigcup_n U_n$ since it is zero on a neighborhood of such that a point. Finally, since $f|U_n = f_n$, $f$ is continuous on $\bigcup_n U_n$ and therefore $f$ is continuous. Since $X$ is pseudocompact, this is a contradiction so $X$ is a $\mathfrak{B}_F$-space. ■

The following result is not a new result (Noble uses this fact in the proof of [[7], Theorem 2.1]) but the author explicitly formulated and proved.

**Proposition 2.9.** *Let $\varphi\colon X \to Y$ be a continuous map of the $\mathfrak{B}_F$-space $X$ onto the space $Y$. Then $Y$ is a $\mathfrak{B}_F$-space.*

**Proof.** Let $V_1, V_2, \ldots$ be a sequence of non-empty open sets in $Y$. For each $n \in \omega$, define $U_n = \varphi^{-1}(V_n)$. By the continuity of $\varphi$, each $U_n$ is open in $X$. Since $X$ is a $\mathfrak{B}_F$-space, there exists a compact set $L \subseteq X$ such that $L \cap U_n \neq \emptyset$ for infinitely many indices $n \in \omega$. Therefore, $\varphi(L)$ is compact and $\varphi(L) \cap V_n \neq \emptyset$ for infinitely many indices $n \in \omega$, i.e. $Y$ is a $\mathfrak{B}_F$-spaces. ■

We have also the following result:

**Theorem 2.10.** *If a space $X$ has a dense subspace $Y$ which is a $\mathfrak{B}_F$-space, then $X$ itself is a $\mathfrak{B}_F$-space.*

**Proof.** Let $V_1, V_2, \ldots$ be a sequence of non-empty open sets in $X$. For each $n \in \omega$, define $U_n = Y \cap V_n$. Then $U_n$ is an open non-empty subset of $Y$. By hypothesis, there exists a compact set $K \subseteq Y$ such that $K \cap U_n \neq \emptyset$ for infinitely many indices $n \in \omega$. Hence, $K \cap V_n \neq \emptyset$ for infinitely many indices and the proof is complete. ■

We finish this preliminary section proving the following result:

**Theorem 2.11.** *Every finite product of $\mathfrak{B}_F$-spaces is a $\mathfrak{B}_F$-space.*

**Proof.** It is enough to prove that if $X, Y$ are $\mathfrak{B}_F$-spaces, then $X \times Y$ is also $\mathfrak{B}_F$-space. Let $W_s = U_s \times V_s$ be non-empty basic open sets in $X \times Y$. Let $K_1 \subseteq X$ be a compact set in $X$ such that $K_1 \cap U_s \neq \emptyset$ for every $s \in L_1$, where $L \subseteq \omega$ and $|L| = \omega$. Let now $K_2 \subseteq Y$ be a compact set in $Y$ such that $K_2 \cap V_s \neq \emptyset$ for every $s \in L_2$, with $L_2 \subseteq L_1$, $|L_2| = \omega$. Therefore $K = K_1 \times K_2$ is compact and satisfies $K \cap W_s \neq \emptyset$ for every $s \in L_2$. The proof is then complete. ■

## 3. Main results.

In this section we prove the two properties of $\mathfrak{B}_F$-spaces mentioned in the introduction which were not proved in the last section.

In the following result we present an alternative proof of the theorem 3.5 enunciated in [3].

**Theorem 3.1.** *Let $X$ be a $\mathfrak{B}_F$-space and let $Y$ be pseudocompact. Then $X \times Y$ is pseudocompact.*

**Proof.** Suppose, on the contrary, that $X\times Y$ is not pseudocompact. By (2.6), there exists an infinite discrete family $U_1, U_2, \dots$ of non-empty open sets in $X \times Y$. Let $\pi\colon X \times Y \to X$ be the projection onto the first factor. There exists an index $n_1 \in \omega, n_1 \geq 2$, such that $\pi(U_1) \cap \pi(U_2) \cap \cdots \cap \pi(U_{n_1}) = \emptyset$; otherwise, there would exist a point $z \in \bigcap_{n=1}^{\infty} \pi(U_n)$ and the set $\{z\} \times Y$ would be a pseudocompact subset of $X \times Y$ which would intersect every $U_n$, a fact which, by (2.6), cannot occur. Pick a minimum $n_1 \in \omega$. Therefore, $\bigcap_{n=1}^{n_1-1} \pi(U_n) \neq \emptyset$. Reasoning in a similar way, we may find a minimum integer $n_2 \geq n_1+2$ such that $\pi(U_{n_1+1}) \cap \cdots \cap \pi(U_{n_2}) = \emptyset$ and continue this process indefinitely. For each $k \in \omega$, $W_k = \pi(U_{n_{k-1}+1}) \cap \cdots \cap \pi(U_{n_k-1})$ is a non-empty subset of $X$. Since $X$ is a $\mathfrak{B}_F$-space, there exists a compact set $K \subseteq X$ such that $K \cap W_k \neq \emptyset$ for infinitely many indices $k$. But then $K \times Y$ is a pseudocompact subset of $X \times Y$ which intersects $U_n$ for infinitely many indices $n \in \omega$, and this is a contradiction. ■

**Theorem 3.2.** *[See [7], Theorem 3.4] Every topological product of $\mathfrak{B}_F$-spaces is pseudocompact.*

**Proof.** Taking only basic open sets in the product, it is enough to consider the case of countably many factors. Suppose then $X_1, X_2, \dots$ is a sequence of $\mathfrak{B}_F$-spaces and let $X = \prod_{n=1}^{\infty} X_n$ be its topological product. Let $W_s = \prod_{n=1}^{\infty} U_n^{(s)}$ be a box in $X$ with non-empty open factors $U_n^{(s)} \subseteq X_n$ and $X_n = U_n^{(s)}$ for almost every $n$. We shall prove that the sequence $\{W_s\colon s \in \omega\}$ cannot be discrete. Assuming it is discrete, we shall reach a contradiction. Let $K_1 \subseteq X_1$ be a compact set such that $K_1 \cap U_1^{(s)} \neq \emptyset$ for every $s \in L_1 \subseteq \omega$, with $|L_1| = \omega$. Let $K_2 \subseteq X_2$ be a compact set such that $K_2 \cap U_2^{(s)} \neq \emptyset$ for every $s \in L_2 \subseteq L_1$, with $|L_2| = \omega$. Continuing this process indefinitely, for each $j \in \omega$ we can find a compact set $K_j \subseteq X_j$ and an infinite subset $L_j$ of $\omega$ such that $K_j \cap U_j^{(s)} \neq \emptyset$ for every $s \in L_j$. We may suppose also that $L_1 \supseteq L_2 \supseteq \cdots$. Let $K = \prod_{j=1}^{\infty} K_j$. The set $K$ is compact by the Tychonoff product theorem. For each $x \in K$, we may find a basic open box $V_x \subseteq X$ such that $V_x \cap W_n \neq \emptyset$ for at most one value of $n$. Since $K$ is compact, we may find a finite union $V$ of the basic sets $V_x$ such that $V \supseteq K$ and $V \cap W_n \neq \emptyset$ for at most finitely many indices $n \in \omega$. The open set $V$ may be expressed in the form:

$$V = L \times \prod_{j=t+1}^{\infty} X_j$$

where $t \in \omega$ and $L$ is an open set in $X_1 \times X_2 \times \cdots \times X_t$ which contains $K_1 \times K_2 \times \cdots \times K_t$. Indeed there is $t$ such that

$$V \supset \prod_{j=1}^{t} K_j \times \prod_{j=t+1} X_j.$$

Hence, for $s \in L_t$ we have

$$V \cap W_s \supseteq \prod_{j=1}^{t} (K_j \cap U_j^{(s)}) \times \prod_{j=t+1}^{\infty} U_j^{(s)} \neq \emptyset$$

This contradiction proves that the sequence $W_1, W_2, \dots$ cannot be discrete and hence $X$ is pseudocompact. ■

We finish this paper with a short proof of a classic result (see [[7], Construction 2.3]):

**Theorem 3.3.** *Every space $X$ is homeomorphic to a closed subspace of a pseudocompact space $Y$.*

**Proof.** We can obviously assume that $X$ is not pseudocompact. For every $z \in \beta X - X$, we define $E_z = \beta X - \{z\}$. We know $E_z$ is locally compact and pseudocompact, and hence, each $E_z$ is a $\mathfrak{B}_F$-space. Taking the diagonal immersion $\varphi$ of $X$ into the product $Y = \prod_{z \in \beta X - X} E_z$, we know $\varphi$ is a homeomorphism of $X$ onto a closed subspace of $Y$. But by (2.8) and (3.2), $Y$ is pseudocompact. ■

In [[1], example 3.4], J. L. Blasco gives an example of a $\mathfrak{B}_F$-space which is not a pseudocompact $k_R$-space (see also [5]).

We finish this note stating the following question are open:

**Question 1.** *Does there exist a non $\mathfrak{B}_F$-space $X$ such that $X \times Y$ is pseudocompact for every pseudocompact space $Y$?*

**Question 2.** *Does every $\mathfrak{B}_F$-space contain a dense subspace which is pseudocompact and $k_R$-space?*

**Question 3.** *Is there a $\mathfrak{B}_F$-space $Y$ which cannot be expressed as the continuous image of a pseudocompact $k_R$-space?*

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