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Sobre polinomios tropicales de una variable

On tropical polynomials of a single variable

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Abstract

In this paper we present a study of the different algebraic properties of the tropical polynomials of a single variable, where a generalization of the tropical semiring $\mathbb{R}_{min}$ is also introduced, which allows to extend certain concepts and gives a successful generalization of the fundamental theorem of tropical algebra.

Keywords: Tropical polynomials, tropical semiring.

Resumen

En este trabajo se presenta un estudio de las diferentes propiedades algebraicas de los polinomios tropicales de una variable, donde se introduce además una generalización del semianillo tropical $\mathbb{R}_{min}$, el cual permite extender ciertos conceptos y dar una generalización exitosa del teorema fundamental del álgebra tropical.

Palabras claves: Polinomios tropicales, semianillo tropical.

1. Introduction

The tropical geometry might be described as a piecewise linear or skeletonized version of algebraic geometry, offering new polyhedral tools to compute invariants of algebraic varieties (see e.g. [1],[2],[**?**]). The algebraic varieties are based on tropical semiring $\mathbb{R}_{min} = \mathbb{R} \cup \{+\infty\}$, with the operations

$$x \odot y = x + y \qquad x \oplus y = min\{x, y\}$$

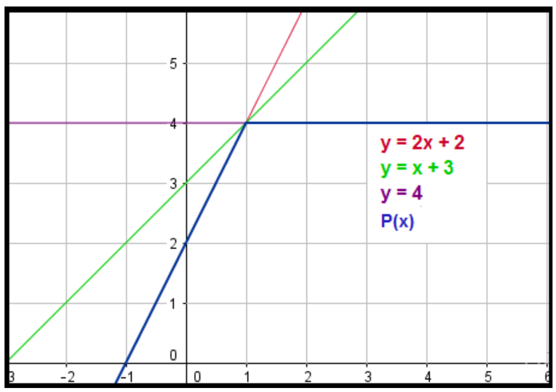
The tropical semiring is idempotent in the sense that $x \oplus x = x$ for any x in $\mathbb{R}_{min}$. It is easy to check that the tropical operations are commutative, associative and satisfy to the distribution law.
The polynomials on $\mathbb{R}_{min}$ satisfy different properties that allow us to handle them suitably and to define a

version of the Fundamental Theorem of Tropical Algebra. In this paper we provide a simple algorithm for factoring tropical polynomials of a single variable and we present a generalization of the tropical semiring $\mathbb{R}_{min}$, which allows to extend certain concepts and gives a successful generalization of the fundamental theorem of tropical algebra .

2. Tropical polynomials

Definition 2.1. *A **tropical polynomial** is a map $p : \mathbb{R}_{min} \to \mathbb{R}_{min}$, of the form $p(x) := a_n \odot x^n \oplus \dots a_1 \odot x \oplus a_0$, where $a_i \in \mathbb{R}_{z_{min}}$ are coefficients of p for $i = 1, 2, ..., n$, and x is the independent variable.*

To understand the algebra of tropical polynomials, it is important to know that there are different polynomials that induce the same function $x \to P(x)$.
For example consider the polynomials $p(x) = 2 \odot x^2 \oplus 3 \odot x \oplus 4$ and $q(x) = 2 \odot x^2 \oplus 4$. Is easy to verify that $p(x) = q(x)$ for any $x \in \mathbb{R}_{min}$, graphically we can see it as



Definition 2.2. *The degree of a tropical polynomial p denoted by $deg(p)$, is the highest degrees of its monomials with non-zero coefficients. The degree of a monomial $a_n x^n$ is n, whenever $a_n \neq 0$.*

Proposition 2.3. *Let p and q be two tropical polynomials such that $p(x) = q(x)$ for all x in $\mathbb{R}_{min}$. Then $deg(p) = deg(q)$.*

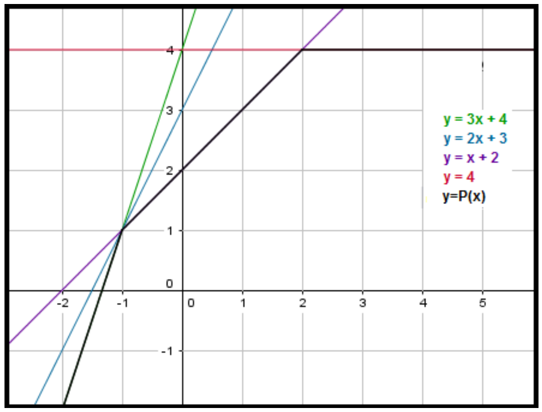
Proof.

$$\begin{aligned} deg(p) &= \lim_{x \to -\infty} \frac{p(x)}{x} \\ &= \lim_{x \to -\infty} \frac{q(x)}{x} \\ &= deg(q) \end{aligned}$$

■

Definition 2.4. *The roots of a tropical polynomial p are the corners of the singular points of the graph of $p(x)$.*

Example 2.5. *We consider the polynomial $p(x) = 4 \odot x^3 \oplus 3 \odot x^2 \oplus 2 \odot x \oplus 4$ and note the corresponding lines.*



The graph of $y = p(x)$ is a piecewise linear function with two corners in $x = -1$ and $x = 2$, is possible to factor $p(x)$ in following way:

$$P(x) = 4 \odot (x \oplus -1)^2 \odot (x \oplus 2)$$

The roots of the polynomial $p(x)$ are defined as the set of corners that appear in the graph of the function of the same polynomial. Let $a = (-1, 1)$ and $b = (2, 4)$ be corners of $p(x)$ and also, let M_a and M_b be matrices defined as:

$$M_a = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad M_b = \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}$$

The columns of the matrices are the directions of the two lines coming out from a and b respectively. Observe that the determinants of these two matrices are exactly the exponents of the two linear forms used in the decomposition of $p(x)$ as a product of linear terms. The following result generalizes the above.

Theorem 2.6. *Any tropical polynomial of degree n factorizes as the product of n linear polynomials.*

Proof.

Let p be a tropical polynomial of degree n. It is clear that the graph generated by this polynomial is composed of lines and segments. Let y_1 and y_2 be consecutive lines in the graph of $p(x)$ with $y_1 = a_j + x_j$ and $y_2 = a_i + x_i$, the corresponding directional matrix for the roots p_{ij} is

$$p(x) = \begin{pmatrix} 1 & 1 \\ i & j \end{pmatrix}$$

For every p_{ij} is associated the factor $\left(x \oplus \frac{x_i - a_j}{j-i}\right)^{j-i}$ with $j_1 < \ldots < j_r$, the indexes of roots of $p(x)$. It is a straightforward calculation (done comparing with the graph of $p(x)$) to see that

$$P(x) = a_n \odot \left(x \oplus \frac{a_{i_j} - a_{i_{r-1}}}{i_r - i_{r-1}}\right)^{i_r - i_{r-1}} \odot \ldots \odot \left(x \oplus \frac{a_{i_2} - a_{i_1}}{i_2 - i_1}\right)^{i_2 - i_1}$$

■

Theorem 2.7. ***(Fundamental theorem of algebra)*** *Let p be a tropical polynomial of degree n, then* $p(x) = a_n \odot (x \oplus x_1)^{n1} \odot ... \odot (x \oplus x_r)^{nr}$ *with* $n_1 + n_2 + ... + n_r = n$ *and* $x_1, x_2, \ldots, x_r$ *are singular points of graphs of p.*

This result is an immediate consequence of the theorem 2.6, taking into account the multiplicity of the roots. Other consequence of this result, is that all polynomial equation has solution in $\mathbb{R}_{min}$.

Corollary 2.8. *$\mathbb{R}_{min}$ is algebraically closed.*

Theorem 2.9. *If $P : \mathbb{R} \to \mathbb{R}$ is a tropical polynomial, then the the following properties are satisfied.*

1. *P is continuous.*
2. *P is piecewise linear, where the number of piecewise is finite.*
3. *P is concave, namely, $P\left(\frac{x+y}{2}\right) \geq \frac{1}{2}(P(x) + P(y))$ for all $x, y \in \mathbb{R}$*

1 and 2 is a immediate consequence of definitions, for 3 we have

$$
\begin{aligned}
P\left(\frac{x+y}{2}\right) &= a_n \odot \left(\frac{x+y}{2}\right)^n \oplus a_{n-1} \odot \left(\frac{x+y}{2}\right)^{n-1} \oplus \ldots \oplus a_0 \\
&= min\{a_n + n\left(\frac{x+y}{2}\right), a_{n-1} + (n-1)\left(\frac{x+y}{2}\right), \ldots, a_0\} \\
&= min\{a_n + \frac{nx+ny}{2}, a_{n-1} + \frac{(n-1)x+(n-1)y}{2}, \ldots, a_0\} \\
&= \frac{1}{2} min\{2a_n + nx + ny, 2a_{n-1} + (n-1)x + (n-1)y, \ldots, 2a_0\} \\
&\geq \frac{1}{2}\{min\{a_n + nx, a_{n-1}(n-1)x, \ldots, a_0\} + min\{a_n + ny, a_{n-1}(n-1)y, \ldots, a_0\} \\
&= \frac{1}{2}[p(x) + p(y)]
\end{aligned}
$$

Each function that satisfy above properties can be represented as the minimum of a finite set of linear functions

3. Semiring of polynomials

In this section we will study the algebraic properties of polynomials, seeing them as elements of a certain tropical semiring. Let $(F, +, \cdot, \leq)$ be an ordered field and $\infty = Sup(F)$. Consider the extension $F_{min} = F \cup \{\infty\}$ and we define the maps

$$
\begin{aligned}
\odot : F_{min} \times F_{min} \to F_{min} \qquad & \oplus : F_{min} \times F_{min} \to F_{min} \\
\odot(x, y) = x + y \qquad & \oplus(x, y) = \left\{\begin{matrix} y & if\ y \leq x \\ x & if\ x \leq y \end{matrix}\right\}
\end{aligned}
$$

It is clear that $\odot$ and $\oplus$ is well defined, it will denote as $\odot(x, y) = x \odot y$ and $\oplus(x, y) = x \oplus y$. It can verified that $(F_{min}, \oplus, \odot)$ is a unital semiring, since $\odot$ is distributive with respect to $\oplus$, which identities are respectively $1_{min} = 0$ y $0_{min} = \infty$.

We will define $F_{min}[x]$ as the set of all equivalence classes of symbols $p(x) = a_n \odot x^n \oplus \ldots a_1 \odot x \oplus a_0$, where $a_n, a_{n-1}, \ldots, a_0$ are elements in F_{min} and $p \sim q$ if and only if $p(x) = q(x)$ for any x in F_{min}. Every element of $F_{min}[x]$ will be denominate as tropical polynomial.

Definition 3.1. *If $p(x) = a_n \odot x^n \oplus \ldots a_1 \odot x \oplus a_0$ and $q(x) = b_m \odot x^m \oplus \ldots b_1 \odot x \oplus b_0$, are elements in $F_{min}[x]$. We will define the tropical sum as $p(x) \oplus q(x) = c_0 \oplus c_1 \odot x \oplus \ldots \oplus c_t x^t$, where $c_i = a_i \oplus b_i$ for $i = 1, 2, ..., t$ and $t = max\{deg(p), deg(q)\}$, moreover $a_i = 0_{min}$ if $i > n$ and $b_i = 0_{min}$ when $i > m$.*

In other words the sum of two tropical polynomials is performed by tropical sum corresponding coefficients of their similar terms, the most complicated operation is the one that we have to define in $F_{min}[x]$, it is the multiplication.

Definition 3.2. *If $p(x) = a_n \odot x^n \oplus \ldots a_1 \odot x \oplus a_0$ and $q(x) = a_m \odot x^m \oplus \ldots a_1 \odot x \oplus a_0$, are elements in $F_{min}[x]$, then of multiplication is defined as $p(x) \odot q(x) = c_0 \oplus c_1 \odot x \oplus \ldots \oplus c_k \odot x^k$, where $c_t = a_t \odot b_0 \oplus a_{t-1} \odot b_1 \oplus \ldots \oplus a_0 \odot b_t$*

Example 3.3. *Let $p(x) = 3 \odot x^3 \oplus 2 \odot x^2 \oplus -1$ and $q(x) = 5 \odot x^4 \oplus 3 \odot x^3 \oplus -1 \odot x \oplus 4$ be elements of $Q_{min}[x]$, we have*

$$\begin{aligned} p(x) \oplus q(x) &= (3 \odot x^3 \oplus 2 \odot x^2 \oplus -1) \oplus (5 \odot x^4 \oplus 3 \odot x^3 \oplus -1 \odot x \oplus 4) \\ &= 5 \odot x^4 \oplus (3 \odot x^3 \oplus 3 \odot x^3) \oplus 2 \odot x^2 \oplus -1 \odot x \oplus (-1 \oplus 4) \\ &= 5 \odot x^4 \oplus 3 \odot x^3 \oplus 2 \odot x^2 \oplus -1 \odot x \oplus -1 \end{aligned}$$

For $p(x) \odot q(x)$ we have:

$$\begin{aligned} c_0 &= -1 \odot 4 = 3 \\ c_1 &= 0_{min} \odot 4 \oplus 2 \odot -1 = 0_{min} \oplus 1 = 1 \\ c_2 &= 2 \odot 4 \oplus 0_{min} \odot -1 \oplus -1 \odot 0_{min} = 6 \oplus 0_{min} \oplus 0_{min} = 6 \\ c_3 &= 3 \odot 4 \oplus 2 \odot -1 \oplus 0_{min} \odot 0_{min} \oplus -1 \odot 3 = 7 \oplus 1 \oplus 0_{min} \oplus 2 = 1 \\ c_4 &= 0_{min} \odot 4 \oplus 3 \odot -1 \oplus 2 \odot 0_{min} \oplus 0_{min} \odot 3 \oplus -1 \odot 5 = 0_{min} \oplus 2 \oplus 0_{min} \oplus 0_{min} \oplus 4 = 2 \end{aligned}$$

Therefore, according to definition,

$$p(x) \odot q(x) = 3 \oplus 1 \odot x \oplus 6 \odot x^2 \oplus 1 \odot x^3 \oplus 2 \odot x^4$$

Thus $(F[x], \oplus, \odot)$ is a semiring, which is denominated **Tropical semiring of polynomials**. The zero of this semiring is given by $0_{min}(x) = 0_{min}$ and unit $1_{min}(x) = 1_{min}$ for all $x \in F_{min}$.

Theorem 3.4. *If $p(x)$ and $q(x)$ are two different elements of $0_{min}(x)$ in $F_{min}[x]$, then $deg(p(x) \odot q(x)) = deg(p(x)) \odot deg(q(x))$.*

Let $p(x) = a_n \odot x^n \oplus \ldots a_1 \odot x \oplus a_0$ and $q(x) = b_m \odot x^m \oplus \ldots b_1 \odot x \oplus b_0$ be elements of $F_{min}[x]$, with a_n and b_m being different to 0_{min}. Hence $deg(p(x)) = n$ and $deg(q(x)) = m$, moreover $p(x) \odot q(x) = c_0 \oplus c_1 \odot x \oplus \ldots \oplus c_k \odot x^k$ where $c_i = a_i \odot b_0 \oplus \ldots \oplus a_0 \odot b_i$. Therefore

$$\begin{aligned} c_{m+n} &= a_{m+n} \odot b_0 \oplus \ldots \oplus a_{m+1} \odot b_{n-1} \oplus a_m \odot b_n \oplus a_{m-1} \odot b_{n+1} \oplus \ldots \oplus a_0 \odot b_{m+n} \\ &= a_m \odot b_n \end{aligned}$$

if $i > m+n$ then the terms c_i are of form $a_j \odot b_{i-j}$, since $i = j + (i-j) > m+n$ then $j > m$ and $i - j > n$ and hence $c_i = 0_{min}$ for every $i > m+n$. It is verified that c_{m+n} is the highest coefficient that is different to 0_{min}. Finally

$deg(p(x) \odot q(x)) = m + n = deg(p(x)) \odot deg(q(x))$

Corollary 3.5. *If $p(x)$ and $q(x)$ are elements of $F_{min}[x]$ wich is different to 0_{min}, then $deg(p(x)) \leq deg(p(x) \odot q(x))$.*

$$\begin{aligned} deg(P(x) \odot Q(x)) &= deg(P(x)) \odot deg(Q(x)) \\ &\geq deg(P(x)) \end{aligned}$$

We generalize certain results obtained in section 2, whose proofs are similar.

Proposition 3.6. *If $p(x)$ y $q(x)$ are elements of $F_{min}[x]$, such that $p(x) = q(x)$ for every x in F_{min}, then $deg(p) = deg(q)$.*

Theorem 3.7. *If F is an Archimedean field, then any tropical polynomial of degree n in $F_{min}[x]$ factorizes as the product of n linear polynomials.*

Given $x, y \in F$, such that $x \leq y$. Archimedean property imply there exist $z \in F$ such that $x \leq z \leq y$, thus we obtain that $p \in \mathbb{F}_{min}[x]$ is composed of lines and segments. Finally this result is concluded by performing a analogous procedure of the Theorem 2.6.

As an immediate consequence of this result it is possible to generalize the fundamental theorem of tropical algebra taking into account the multiplicity of the roots, which is enunciated below.

Theorem 3.8. *Let F be an Archimedean field and p a tropical polynomial of degree n en $F_{min}[x]$, then $p(x) = a_n \odot (x \oplus x_1)^{n1} \odot ... \odot (x \oplus x_r)^{nr}$ with $n_1 + n_2 + ... + n_r = n$ and $x_1, x_2, \ldots, x_r$ are singular points of graph of p.*

Referencias

[1] Erwan Brugalle and Kristin Shaw. A bit of tropical geometry. The American Mathematical Monthly, 121(7):pp. 563-589, 2014.

[2] D. Maclagan, B. Sturmfels. Introduction to Tropical Geometry. University of California.(2015).Pag 15-38.

[3] N. Grigg y N. Manwaring, An Elementary Proof of the Fundamental Theorem of Tropical Algebra, arXiv:0707.2591v1.

[4] T. Theobald, On the frontiers of polynomial computations in tropical geometry. Pag 1-6 (2005)

[5] J. Richter-Gebert, B. Sturmfels, y T. Theobald, First steps in tropical geometry, Idempotent mathematics and mathematical physics, Contemp. Math., 377, Amer. Math. Soc., Providence, RI, 289-317 (2005).