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Some integral inequalities involving the $k-$Beta function and generalized convex stochastic processes.

Algunas desigualdades integrales que involucran la función $k-$Beta y procesos estocásticos convexos generalizados.

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Abstract

In the present work some integral inequalities that involve the $k-$Beta function and stochastic processes whose absolute values posses the property of convexity, or $P-$convexity, $s-$convexity in the second sense or $(m,h_1,h_2)-$convexity are established. Similarly, some others integral inequalities for stochastic processes whose $r-$th powers of its absolute values posses these kind of generalized convexity are established making use of the Hölder's inequality and power mean inequality.

Keywords: Integral inequalities, $k-$Beta function, Generalized convex Stochastic Processes
2015 MSC: 60E15, 26B25, 26A33

Resumen

En el presente trabajo se establecen algunas desigualdades integrales que involucran la función k-Beta y procesos estocásticos cuyos valores absolutos poseen la propiedad de convexidad, o $P-$convexidad, $s-$convexidad en segundo sentido o $(m,h_1,h_2)-$convexidad. Del mismo modo, se encuentran otras desigualdades integrales para procesos estocásticos cuyas r-ésimas potencias de sus valores absolutos poseen este tipo de convexidad generalizada haciendo uso de la desigualdad de Hölder y la desigualdad de media de potencias.

Palabras claves: Desigualdades integrales, Función $k-$Beta , Procesos estocástios convexos generalizados
2015 MSC: 60E15, 26B25, 26A33

1. Introduction

Convexity is a basic notion in geometry, but it is also widely used in other areas of mathematics. The use of techniques of convexity appears in many branches of mathematics and sciences, such as Theory of Optimization and Theory of Inequalities, Functional Analysis, Mathematical Programming and Game Theory, Theory of Numbers, Variational Calculus and its interrelation with these branches shows itself day by day deeper and fruitful.

A function $f : I \to \mathbb{R}$ is said to be convex if for all $x, y \in I$ and $t \in [0, 1]$ the inequality

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$$

holds. Over time, several problems and applications have arisen, and these have given rise to generalizations of the concept of convex function, and also numerous works of investigation have been realized extending results on inequalities for this kind of convexity: quasi-convexity [17], $s-$convexity in the first and second sense [2], logarithmically convexity [1], $m-$convex [14], $(s, \eta)-$convex [24] and others.

The study on convex stochastic processes began in 1974 when B. Nagy applied a characterization of measurable stochastic processes to solving a generalization of the (additive) Cauchy functional equation [12]. In 1980, K. Nikodem considered convex stochastic processes [13]. In 1995,A. Skowronski obtained some further results on convex stochastic processes, which generalize some known properties of convex functions [21]. From that moment many researchers began to merge the properties of generalized convexity with the stochastic processes. By example, in the year 2014, E. Set et. al. investigated Hermite-Hadamard type inequalities for $s-$convex stochastic processes in the second sense [16], in 2015 M. Tomar et. al. worked on log-convex stochastic processes [23], recently, in 2018, the author introduced the concept of $(m, h_1, h_2)-$convex stochastic processes and related it to some inequalities for fractional integrals [8]. For other results related to stochastic processes see [3],[5],[7],[11],[18] and [19], where further references are given.

Following this line of research, the present work aims to find some integral inequalities that involve the $k-$Beta function and the stochastic processes which absolute value are convex, $P-$convex, $s-$convex in the second sense or $(m, h_1, h_2)-$convex.

2. Preliminaries

The following notions corresponds to ordinary and convex Stochastic Process. References about it can be found in [9, 10, 11, 20, 21].

Definition 2.1. *Let $(\Omega, \mathcal{A}, \mu)$ be an arbitrary probability space. A function $X : \Omega \to \mathbb{R}$ is called a random variable if it is $\mathcal{A}$-measurable. Let $I \subset \mathbb{R}$ be time. A collection of random variable $X(t, w), t \in I$ with values in $\mathbb{R}$ is called a stochastic processes.*

1. *If $X(t, w)$ takes values in $S = \mathbb{R}^d$ if is called vector-valued stochastic process.*
2. *If the time I is a discrete subset of $\mathbb{R}$, then $X(t, w)$ is called a discrete time stochastic process.*
3. *If the time I is an interval in $\mathbb{R}$, it is called a stochastic process with continuous time.*

Definition 2.2. *Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and $I \subset \mathbb{R}$ be an interval. We say that the stochastic process $X : I \times \Omega \to \mathbb{R}$ is called*

1. *Continuous in probability in the interval I if for all $t_0 \in I$ we have*

$$\mu - \lim_{t \to t_0} X(t, \cdot) = X(t_0, \cdot),$$

where $\mu - \lim$ denotes the limit in probability.

2. *Mean-square continuous in the interval I if for all $t_0 \in I$*

$$\mu - \lim_{t \to t_0} \mathbb{E}(X(t,\cdot) - X(t_0,\cdot)) = 0,$$

where $\mathbb{E}(X(t,\cdot))$ denote the expectation value of the random variable $X(t,\cdot)$.

3. *Increasing (decreasing) if for all $u, v \in I$ such that $t < s$,*

$$X(u,\cdot) \leq X(v,\cdot), \qquad (X(u,\cdot) \geq X(v,\cdot)).$$

4. *Monotonic if it's increasing or decreasing.*
5. *Differentiable at a point $t \in I$ if there is a random variable $X'(t,\cdot) : I \times \Omega \to \mathbb{R}$, such that*

$$X'(t,\cdot) = \mu - \lim_{t \to t_0} \frac{X(t,\cdot) - X(t_0,\cdot)}{t - t_0}.$$

We say that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is continuous (differentiable) if it is continuous (differentiable) at every point of the interval I (See [9], [11],[21]).

Definition 2.3. *Let $(\Omega, \mathcal{A}, \mu)$ be a probability space $I \subset \mathbb{R}$ be an interval with $E(X(t)^2) < \infty$ for all $t \in I$. Let $[a,b] \subset I, a = t_0 < t_1 < ... < t_n = b$ be a partition of $[a,b]$ and $\theta_k \in [t_{k-1}, t_k]$ for $k = 1, 2, ..., n$. A random variable $Y : \Omega \to \mathbb{R}$ is called mean-square integral of the process $X(t,\cdot)$ on $[a,b]$ if the following identity holds:*

$$\lim_{n \to \infty} E\left[\sum_{k=1}^{\infty} X(\theta_k,\cdot)(t_k - t_{k-1}) - Y(\cdot)\right]^2 = 0,$$

then we can write

$$\int_a^b X(t,\cdot)dt = Y(\cdot) \qquad (a.e.).$$

Also, mean square integral operator is increasing, that is,

$$\int_a^b X(t,\cdot)dt \leq \int_a^b Z(t,\cdot)dt \qquad (a.e.)$$

where $X(t,\cdot) \leq Z(t,\cdot)$ in $[a,b]$.

For further reading on stochastic calculus, reader may refer to [4], [19] and [22].

The following definition can be found in the works of D. Kotrys [10], E. Set [16] and A. Skowronski [20].

Definition 2.4. *Set $(\Omega, \mathcal{A}, P)$ be a probability space and $I \subset \mathbb{R}$ be an interval. We say that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is*

1. *Convex if the inequality*

$$X(\lambda u + (1-\lambda)v,\cdot) \leq \lambda X(u,\cdot) + (1-\lambda)X(v,\cdot) \tag{1}$$

holds almost everywhere for all $u, v \in I$ and $\lambda \in [0,1]$.

2. *P−convex if the inequality*
$$X(\lambda u + (1-\lambda)v, \cdot) \leq X(u, \cdot) + X(v, \cdot) \tag{2}$$
holds almost everywhere for all $u, v \in I$ *and* $\lambda \in [0,1]$
3. *s−convex in the second sense if the inequality*
$$X(\lambda u + (1-\lambda)v, \cdot) \leq \lambda^s X(u, \cdot) + (1-\lambda)^s X(v, \cdot) \tag{3}$$
holds almost everywhere for all $u, v \in I$ *and* $\lambda \in [0,1]$ *and for some fixed* $s \in (0,1]$.

This is one of the basis for the development of this work.

Definition 2.5. *[8] Let* $h_1, h_2 : [0,1] \to \mathbb{R}$ *be a non negative functions, where* $h_1, h_2 \not\equiv 0$*, and* $m \in (0,1]$ *. We say that a stochastic process* $X : I \times \Omega \to \mathbb{R}$ *is a* (m, h_1, h_2)*−convex if*

$$X(ta + m(1-t)b, \cdot) \leq h_1(t)X(a, \cdot) + mh_2(t)X(b, \cdot) \qquad (a.e.)$$

for all $a, b \in I$ *and* $t \in [0,1]$ *.*

Also, in the development of this work we use the $k-$Beta function and it is useful recall some notes about it. From the work of R. Diaz y E. Pariguan [6] it is extracted the following.

Definition 2.6. *Let* $x \in \mathbb{C}, k \in \mathbb{R}$ *and* $n \in \mathbb{N}^+$ *For* $k > 0$*, the Pochhammer k−symbol is given by*

$$(x)_{n,k} = x(x+k)(x+2k)\cdots(x+(n-1)k).$$

Definition 2.7. *For* $k > 0$*, the k−Gamma function* Γ_k *is given by*

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}} \qquad x \in \mathbb{C} \setminus k\mathbb{Z}^-.$$

Definition 2.8. *The k−Beta function* $B_k(x,y)$ *is given by*

$$B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)} \qquad Re(x) > 0, Re(y) > 0.$$

Also, the same authors established an integral representation for the $k-$Beta function as follow [6, Proposition 14]:

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt,$$

also a property follows from the definition, as it can be sawn in [15]:

$$B_k(x+k,y) = \frac{x}{x+y}B_k(x,y) \text{ and } B_k(x,y+k) = \frac{y}{x+y}B_k(x,y). \tag{4}$$

Some others properties of the $k-$Beta functions, and also for $k-$Beta function with several variables, can be found in the work of M. Rehman et. al. [15].

With these notions it is presented the main results of this work.

3. Main Results

Lemma 3.1. *Let $X : I \times \Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Then the equality*

$$\int_a^b (u-a)^{p/k} (b-u)^{q/k} X(u,\cdot)du = (b-a)^{\frac{p+q}{k}+1} \int_0^1 (1-t)^{p/k} t^{q/k} X(ta+(1-t)b,\cdot)\,dt$$

holds for some fixed $p, q, k > 0$.

Proof. Let $u = ta + (1-t)b$. Then $t = (b-u)/(b-a)$, $1 - t = (u-a)/(b-a)$ and $dt = -du/(b-a)$, so

$$\int_0^1 (1-t)^{p/k} t^{q/k} X(ta+(1-t)b,\cdot)\,dt = \frac{1}{(b-a)^{\frac{p}{k}+\frac{q}{k}+1}} \int_a^b (u-a)^{p/k} (b-u)^{q/k} X(u,\cdot)\,du.$$

The proof is complete. ■

The following results for stochastic processes whose absolute values are convex, including $r-$th powers of them, are established.

Theorem 3.2. *Let $X : I \times \Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Let $p, q, k > 0$, if $|X|$ is convex on $[a, b]$, where $a, b \in I$ and $a < b$, then the following inequality holds almost everywhere*

$$\int_a^b (u-a)^{p/k} (b-u)^{q/k} X(u,\cdot)du \le (b-a)^{\frac{p+q}{k}+1} \frac{kB_k(p,q)}{(p+q)_{3,k}} \Big((q)_{2,k}\, p\, |X(a,\cdot)| + (p)_{2,k}\, q\, |X(b,\cdot)|\Big). \quad (5)$$

Proof. Using Lemma 3.1, the convexity of $|X|$, the definition of the $k-$Beta function and the property (4), we have

$$\begin{aligned}
&\int_a^b (u-a)^{p/k} (b-u)^{q/k} X(u,\cdot)du \\
&\le (b-a)^{\frac{p+q}{k}+1} \int_0^1 (1-t)^{p/k} t^{q/k} |X(ta+(1-t)b,\cdot)|\,dt \\
&\le (b-a)^{\frac{p+q}{k}+1} \int_0^1 (1-t)^{p/k} t^{q/k} (t|X(a,\cdot)| + (1-t)|X(b,\cdot)|)\,dt \\
&= (b-a)^{\frac{p+q}{k}+1} \left(|X(a,\cdot)| \int_0^1 (1-t)^{p/k} t^{q/k+1} dt + |X(b,\cdot)| \int_0^1 (1-t)^{p/k+1} t^{q/k} dt\right) \\
&= (b-a)^{\frac{p+q}{k}+1} k\left(B_k(p+k, q+2k)|X(a,\cdot)| + B_k(p+2k, q+k)|X(b,\cdot)|\right) \\
&= (b-a)^{\frac{p+q}{k}+1} kB_k(p,q)\left(\frac{(q)_{2,k}\, p}{(p+q)_{3,k}} |X(a,\cdot)| + \frac{(p)_{2,k}\, q}{(p+q)_{3,k}} |X(b,\cdot)|\right)
\end{aligned}$$

The proof is complete. ■

Theorem 3.3. *Let $X : I \times \Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Let $p, q, k > 0$, if $|X|^r$ is convex on $[a, b]$ for $r > 1$, where $a, b \in I$ and $a < b$, then the following inequality holds almost everywhere*

$$\int_a^b (u-a)^{p/k} (b-u)^{q/k} X(u,\cdot)du \tag{6}$$

$$\leq 2^{-1/r} (b-a)^{\frac{p+q}{k}+1} \left(\frac{kpq}{(lp+lq)_{2,k}} B_k(lp, lq) \right)^{1/l} \left(|X(a,\cdot)|^r + |X(b,\cdot)|^r \right)^{1/r},$$

where $(1/l) + (1/r) = 1$.

Proof. From Lemma 3.1 and using the Hölder inequality we have

$$\begin{aligned} &\int_a^b (u-a)^{p/k} (b-u)^{q/k} X(u,\cdot)du \\ &\leq (b-a)^{\frac{p+q}{k}+1} \int_0^1 (1-t)^{p/k} t^{q/k} |X(ta+(1-t)b,\cdot)| dt \\ &\leq (b-a)^{p+q+1} \left(\int_0^1 (1-t)^{lp/k} t^{lq/k} dt \right)^{1/l} \left(\int_0^1 |X(ta+(1-t)b,\cdot)|^r dt \right)^{1/r}. \end{aligned} \tag{7}$$

Since $|X|^r$ is a convex stochastic process then

$$\int_0^1 |X(ta+(1-t)b,\cdot)|^r dt \leq \int_0^1 t|X(a,\cdot)|^r + (1-t)|X(b,\cdot)|^r dt \tag{8}$$

$$= \frac{|X(a,\cdot)|^r + |X(b,\cdot)|^r}{2},$$

and using the definition of the k−Beta function and the property (4), we get

$$\begin{aligned} \int_0^1 (1-t)^{lp/k} t^{lq/k} dt &= kB_k(lp+k, lq+k) \\ &= k\frac{pq}{(lp+lq)_{2,k}} B_k(lp, lq) \end{aligned} \tag{9}$$

So replacing (8) and (9) in (7) it is attained the required inequality (6).

The proof is complete. ■

Theorem 3.4. *Let $X : I \times \Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Let $p, q, k > 0$, if $|X|^r$ is convex on $[a, b]$ for $r > 1$, where $a, b \in I$ and $a < b$, then the following inequality holds almost everywhere*

$$\begin{aligned} &\int_a^b (u-a)^{p/k} (b-u)^{q/k} X(u,\cdot)du \\ &\leq k(b-a)^{\frac{p+q}{k}+1} \left[\frac{pq}{(p+q)_{2,k}} B_k(p,q) \right]^{1-1/r} \times \\ &\qquad \left(\frac{(q)_{2,k}\, p}{(p+q)_{3,k}} B_k(p,q) |X(a,\cdot)|^r + \frac{(p)_{2,k}\, q}{(p+q)_{3,k}} B_k(p,q) |X(b,\cdot)|^r \right)^{1/r}. \end{aligned} \tag{10}$$

Proof. From Lemma 3.1 and using the power mean inequality for $r \geq 1$ we have

$$\begin{aligned}
&\int_a^b (u-a)^{p/k}(b-u)^{q/k} X(u,\cdot)du \\
&\leq (b-a)^{\frac{p+q}{k}+1} \int_0^1 (1-t)^{p/k} t^{q/k} |X(ta+(1-t)b,\cdot)|\,dt \\
&\leq (b-a)^{\frac{p+q}{k}+1} \left(\int_0^1 (1-t)^{p/k} t^{q/k} dt \right)^{1-1/r} \times \\
&\qquad \left(\int_0^1 (1-t)^{p/k} t^{q/k} |X(ta+(1-t)b,\cdot)|^r\,dt \right)^{1/r}.
\end{aligned}$$

Making use of the convexity of the stochastic process $|X|^r$ and the definition the $k-$Beta function, we get

$$\begin{aligned}
&\int_0^1 (1-t)^{p/k} t^{q/k} |X(ta+(1-t)b,\cdot)|^r\,dt \\
&\leq \int_0^1 (1-t)^{p/k} t^{q/k} \left(t\,|X(a,\cdot)|^r + (1-t)\,|X(b,\cdot)|^r\right) dt \\
&= k\frac{(q)_{2,k}\,p}{(p+q)_{3,k}} B_k(p,q)\,|X(a,\cdot)|^r + k\frac{(p)_{2,k}\,q}{(p+q)_{3,k}} B_k(p,q)\,|X(b,\cdot)|^r .
\end{aligned} \tag{11}$$

Replacing (11) in the previous inequality it is attained the desired inequality (10).

The proof is complete. ■

The following results for stochastic processes whose absolute values are $P-$convex, including $r-$th powers of them, are established.

Theorem 3.5. *Let $X : I \times \Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Let $p,q,k > 0$, if $|X|$ is $P-$convex on $[a,b]$ where $a,b \in I$ and $a < b$, then the following inequality holds almost everywhere*

$$\begin{aligned}
&\int_a^b (u-a)^{p/k}(b-u)^{q/k} X(u,\cdot)du \\
&\leq (b-a)^{\frac{p+q}{k}+1} \frac{kpq}{(p+q)_{2,k}} B_k(p,q)\left(|X(a,\cdot)| + |X(b,\cdot)|\right).
\end{aligned} \tag{12}$$

Proof. Using Lemma 3.1 , the definition of the $k-$Beta function and the $P-$convexity of $|X|$, we have

$$\begin{aligned}
&\int_a^b (u-a)^{p/k}(b-u)^{q/k} X(u,\cdot)du \\
&\leq (b-a)^{\frac{p+q}{k}+1} \int_0^1 (1-t)^{p/k} t^{q/k} |X(ta+(1-t)b,\cdot)|\,dt \\
&\leq (b-a)^{\frac{p+q}{k}+1} \int_0^1 (1-t)^{p/k} t^{q/k} \left(|X(a,\cdot)| + |X(b,\cdot)|\right) dt \\
&= \left(|X(a,\cdot)| + |X(b,\cdot)|\right)(b-a)^{\frac{p+q}{k}+1} \int_0^1 (1-t)^{p/k} t^{q/k} dt \\
&= k\,(b-a)^{\frac{p+q}{k}+1} \frac{pq}{(p+q)_{2,k}} B_k(p,q)\left(|X(a,\cdot)| + |X(b,\cdot)|\right).
\end{aligned}$$

The proof is complete. ■

Theorem 3.6. *Let $X : I \times \Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Let $p, q, k > 0$, if $|X|^r$ is $P-$convex on $[a, b]$ for $r > 1$, where $a, b \in I$ and $a < b$, then the following inequality holds almost everywhere*

$$\int_a^b (u-a)^{p/k} (b-u)^{q/k} X(u,\cdot)du \tag{13}$$

$$\leq (b-a)^{\frac{p+q}{k}+1} \left[\frac{kpq}{(lp+lq)_{2,k}} B_k(lp, lq) \right]^{1/l} \left(|X(a,\cdot)|^r + |X(b,\cdot)|^r\right)^{1/r},$$

where $(1/l) + (1/r) = 1$.

Proof. From Lemma 3.1 and using the Hölder inequality we have

$$\int_a^b (u-a)^{p/k} (b-u)^{q/k} X(u,\cdot)du \tag{14}$$

$$\leq (b-a)^{\frac{p+q}{k}+1} \int_0^1 (1-t)^{p/k} t^{q/k} |X(ta+(1-t)b,\cdot)| dt$$

$$\leq (b-a)^{\frac{p+q}{k}+1} \left(\int_0^1 (1-t)^{lp/k} t^{lq/k} dt \right)^{1/l} \left(\int_0^1 |X(ta+(1-t)b,\cdot)|^r dt \right)^{1/r}.$$

Since $|X|^r$ is $P-$convex Stochastic process then

$$\int_0^1 |X(ta+(1-t)b,\cdot)|^r dt \leq |X(a,\cdot)|^r + |X(b,\cdot)|^r, \tag{15}$$

and using the definition of the $k-$Beta function we get

$$\begin{aligned} \int_0^1 (1-t)^{lp/k} t^{lq/k} dt &= kB_k(lp+k, lq+k) \\ &= \frac{kpq}{(lp+lq)_{2,k}} B_k(lp, lq) \end{aligned} \tag{16}$$

So replacing (15) and (16) in (14) it is attained the required inequality (13).

The proof is complete. ■

Theorem 3.7. *Let $X : I \times \Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Let $p, q, k > 0$, if $|X|^r$ is $P-$convex on $[a, b]$ for $r > 1$, where $a, b \in I$ and $a < b$, then the following inequality holds almost everywhere*

$$\int_a^b (u-a)^{p/k} (b-u)^{q/k} X(u,\cdot)du \tag{17}$$

$$\leq (b-a)^{\frac{p+q}{k}+1} \frac{kpq}{(p+q)_{2,k}} B_k(p, q) \left(|X(a,\cdot)|^r + |X(b,\cdot)|^r\right)^{1/r}.$$

Proof. From Lemma 3.1 and using the power mean inequality for $r \geq 1$ and the $P-$convexity of $|X|^r$ we have

$$\int_a^b (u-a)^{p/k} (b-u)^{q/k} X(u,\cdot)du$$

$$\leq (b-a)^{\frac{p+q}{k}+1}\int_0^1 (1-t)^{p/k}\, t^{q/k}\, |X\,(ta+(1-t)b,\cdot)|\, dt$$

$$\leq (b-a)^{\frac{p+q}{k}+1}\left(\int_0^1 (1-t)^{p/k}\, t^{q/k}\right)^{1-1/r}\left(\int_0^1 (1-t)^{p/k}\, t^{q/k}\, |X\,(ta+(1-t)b,\cdot)|^r\, dt\right)^{1/r}$$

$$\leq (b-a)^{\frac{p+q}{k}+1}\left(\int_0^1 (1-t)^{p/k}\, t^{q/k}\right)^{1-1/r}\times$$

$$(|X(a,\cdot)|^r+|X\,(b,\cdot)|^r)^{1/l}\left(\int_0^1 (1-t)^{p/k}\, t^{q/k} dt\right)^{1/r}$$

$$= k\,(b-a)^{\frac{p+q}{k}+1}\,\frac{pq}{(p+q)_{2,k}}B_k(p,q)\,(|X(a,\cdot)|^r+|X\,(b,\cdot)|^r)^{1/r}\,.$$

The proof is complete. ■

The following results for stochastic processes whose absolute values are $s-$convex in the second sense, including $r-$th powers of them, are established.The following results are established for $s-$convex stochastic processes.

Theorem 3.8. *Let $X : I\times\Omega\to\mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Let $p,q,k>0$, if $|X|$ is $s-$convex in the second sense on $[a,b]$ for some $s\in(0,1]$, where $a,b\in I$ and $a<b$, then the following inequality holds almost everywhere*

$$\int_a^b (u-a)^{p/k}\,(b-u)^{q/k}\, X(u,\cdot)du \tag{18}$$

$$\leq k\,(b-a)^{\frac{p+q}{k}+1}\,(I_1 B_k\,(p,,q+ks))\,|X\,(a,\cdot)|+I_2 B_k\,(p+ks,q)\,|X\,(b,\cdot)|)\,.$$

where

$$I_1=\frac{p(q+ks)}{(p+q+ks)_{2,k}}\ \ and\ \ I_2=\frac{(p+ks)q}{(p+q+ks)_{2,k}}.$$

Proof. Using Lemma 3.1, the definition of the $k-$Beta function and the $s-$convexity of $|X|$, we have

$$\begin{aligned}
&\int_a^b (u-a)^{p/k}\,(b-u)^{q/k}\, X(u,\cdot)du\\
\leq\ & (b-a)^{\frac{p+q}{k}+1}\int_0^1 (1-t)^{p/k}\, t^{q/k}\, |X\,(ta+(1-t)b,\cdot)|\, dt\\
\leq\ & (b-a)^{\frac{p+q}{+}1}\int_0^1 (1-t)^{p/k}\, t^{q/k}\,(t^s\,|X\,(a,\cdot)|+(1-t)^s\,|X\,(b,\cdot)|)\, dt\\
\leq\ & (b-a)^{\frac{p+q}{k}+1}\left(|X\,(a,\cdot)|\int_0^1 (1-t)^{p/k}\, t^{\frac{q}{k}+s}dt+|X\,(b,\cdot)|\int_0^1 (1-t)^{\frac{p}{k}+s}\, t^q dt\right)\\
=\ & k\,(b-a)^{\frac{p+q}{k}+1}\,(|X\,(a,\cdot)|\,B_k\,(p+k,q+k(s+1))+|X\,(b,\cdot)|\,B_k\,(p+k(s+1),q+k))\,,
\end{aligned}$$

then applying the property (4), we obtain the desired result.

The proof is complete. ■

Theorem 3.9. *Let $X : I \times \Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Let $p, q, k > 0$, if $|X|^r$ is $s-$convex in the second sense on $[a, b]$ for $r > 1$ and some $s \in (0, 1]$, where $a, b \in I$ and $a < b$, then the following inequality holds almost every where*

$$\begin{aligned} &\int_a^b (u-a)^{p/k}(b-u)^{q/k} X(u,\cdot)du \qquad (19)\\ &\leq (b-a)^{\frac{p+q}{k}+1}(s+1)^{-1/r}\left[\frac{kpq}{(lp+lq)_{2,k}}B_k(lp,lq)\right]^{1/l}\left(|X(a,\cdot)|^r+|X(b,\cdot)|^r\right)^{1/r}\end{aligned}$$

where $(1/l) + (1/r) = 1$.

Proof. From Lemma 3.1 and using the Hölder inequality we have

$$\begin{aligned} &\int_a^b (u-a)^{p/k}(b-u)^{q/k} X(u,\cdot)du \qquad (20)\\ &\leq (b-a)^{\frac{p+q}{k}+1}\int_0^1 (1-t)^{p/k} t^{q/k} \left|X(ta+(1-t)b,\cdot)\right| dt\\ &\leq (b-a)^{\frac{p+q}{k}+1}\left(\int_0^1 (1-t)^{lp/k} t^{lq/k}dt\right)^{1/l}\left(\int_0^1 \left|X(ta+(1-t)b,\cdot)\right|^r dt\right)^{1/r}.\end{aligned}$$

Since $|X|^r$ is $s-$convex stochastic process in the second sense then

$$\begin{aligned} &\int_0^1 \left|X(ta+(1-t)b,\cdot)\right|^r dt \qquad (21)\\ &\leq |X(a,\cdot)|^r\int_0^1 t^s dt+|X(b,\cdot)|^r\int_0^1(1-t)^s dt\\ &= \frac{|X(a,\cdot)|^r+|X(b,\cdot)|^r}{s+1},\end{aligned}$$

and using the definition of the $k-$Beta function we get

$$\begin{aligned}\int_0^1 (1-t)^{lp/k} t^{lq/k}dt &= kB_k(lp+k,lq+k)\\ &= k\frac{pq}{(lp+lq)_{2,k}}B_k(lp,lq) \qquad (22)\end{aligned}$$

So, replacing (21) and (22) in (20) it is attained the desired inequality (19).

The proof is complete. ■

Theorem 3.10. *Let $X : I \times \Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Let $p, q, k > 0$, if $|X|^r$ is $s-$convex in the second sense on $[a, b]$ for $r > 1$ and $s \in (0, 1]$, where $a, b \in I$ with $a < b$, then the following inequality holds almost everywhere*

$$\begin{aligned} &\int_a^b (u-a)^{p/k}(b-u)^{q/k} X(u,\cdot)du \qquad (23)\\ &\leq k(b-a)^{\frac{p+q}{k}+1}\left(\frac{pq}{(p+q)_{2,k}}B_k(p,q)\right)^{1-1/r}\times\\ &\left(|X(a,\cdot)|^r\frac{p(q+ks)}{(p+q+ks)_{2,k}}B_k(p,q+ks)+|X(b,\cdot)|^r\frac{(p+ks,q)}{(p+q+ks)_{2,k}}B_k(p+ks,q)\right)^{1/r}\end{aligned}$$

Proof. From Lemma 3.1 and using the power mean inequality for $r \geq 1$ we have

$$\int_a^b (u-a)^{p/k} (b-u)^{q/k} X(u,\cdot)du \tag{24}$$

$$\leq (b-a)^{\frac{p+q}{k}+1} \int_0^1 (1-t)^{p/k} t^{q/k} \left| X(ta+(1-t)b,\cdot) \right| dt$$

$$\leq (b-a)^{\frac{p+q}{k}+1} \left(\int_0^1 (1-t)^{p/k} t^{q/k} dt \right)^{1-1/r} \times$$

$$\left(\int_0^1 (1-t)^{p/k} t^{q/k} \left| X(ta+(1-t)b,\cdot) \right|^r dt \right)^{1/r}.$$

Since $|X|^r$ is $s-$convex in the second sense and using the definition of the $k-$Beta function we get

$$\int_0^1 (1-t)^{p/k} t^{q/k} \left| X(ta+(1-t)b,\cdot) \right|^r dt$$

$$\leq \int_0^1 (1-t)^{p/k} t^{q/k} \left(t^s |X(a,\cdot)|^l + (1-t)^s |X(b,\cdot)|^r \right) dt$$

$$\leq |X(a,\cdot)|^r \int_0^1 (1-t)^{p/k} t^{\frac{q}{k}+s} dt + |X(b,\cdot)|^r \int_0^1 (1-t)^{\frac{p}{k}+s} t^{q/k} dt$$

$$= k \left(|X(a,\cdot)|^r B_k(p+k, q+k(s+1)) + |X(b,\cdot)|^r B_k(p+k(s+1), q+k) \right).$$

With this last result and again using the definition of the $k-$Beta function and the property (4) in the inequality (31) we obtain

$$\int_a^b (u-a)^{p/k} (b-u)^{q/k} X(u,\cdot)du$$

$$\leq k(b-a)^{\frac{p+q}{k}+1} \left(\frac{pq}{(p+q)_{2,k}} B_k(p,q) \right)^{1-1/r} \times$$

$$\left(|X(a,\cdot)|^r \frac{p(q+ks)}{(p+q+ks)_{2,k}} B_k(p,q+ks) + |X(b,\cdot)|^r \frac{(p+ks,q)}{(p+q+ks)_{2,k}} B_k(p+ks,q) \right)^{1/r}$$

The proof is complete. ■

The following results for stochastic processes whose absolute values are $(m.h_1,h_2)-$convex, including $r-$th powers of them, are established.

Theorem 3.11. *Let $h_1, h_2 : [0,1] \to \mathbb{R}$ be a non negative functions, $m \in (0,1]$ and $X : I \times \Omega \to \mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Let $p,q,k > 0$, if $|X|$ is $(m,h_1,h_2)-$convex on $[a,b]$, where $a,b \in I$ and $a < b$, then the following inequality holds almost everywhere*

$$\int_a^b (u-a)^{p/k} (b-u)^{q/k} X(u,\cdot)du \tag{25}$$

$$\leq (b-a)^{\frac{p+q}{k}+1}\left(|X(a,\cdot)|\, I(h_1)+|X(b,\cdot)|\, I(h_2)\right),$$

where

$$I(h_1)=\int_0^1 (1-t)^{p/k}\, t^{q/k} h_1(t)dt$$

and

$$I(h_2)=\int_0^1 (1-t)^{p/k}\, t^{q/k} h_2(t)dt.$$

Proof. Using Lemma 3.1, the definition of the $k-$Beta function and the $(m,h_1,h_2)-$convexity of $|X|$, we have

$$\begin{aligned}
&\int_a^b (u-a)^{p/k}(b-u)^{q/k} X(u,\cdot)du \\
&\quad\leq\quad (b-a)^{\frac{p+q}{k}+1}\int_0^1 (1-t)^{p/k}\, t^{q/k}\, |X(ta+(1-t)b,\cdot)|\, dt \\
&\quad\leq\quad (b-a)^{\frac{p+q}{+}1}\int_0^1 (1-t)^{p/k}\, t^{q/k}\left(h_1(t)\,|X(a,\cdot)|+mh_2(t)\,|X(b,\cdot)|\right)dt \\
&\quad=\quad (b-a)^{\frac{p+q}{k}+1}\left(|X(a,\cdot)|\, I(h_1)+|X(b,\cdot)|\, I(h_2)\right),
\end{aligned}$$

where

$$I(h_1)=\int_0^1 (1-t)^{p/k}\, t^{q/k} h_1(t)dt$$

and

$$I(h_2)=\int_0^1 (1-t)^{p/k}\, t^{q/k} h_2(t)dt.$$

The proof is complete. ■

Remark 3.12. *If in Theorem 3.11 we choose $m=1, h_1(t)=t$ and $h_2(t)=1-t$ for $t\in[0,1]$ then we obtain the inequality* (5) *in Theorem 3.2 for convex stochastic processes. Similarly, if we choose $m=1, h_1(t)=1$ and $h_2(t)=1$ for $t\in[0,1]$ then we get the inequality* (12) *in Theorem 3.5 for $P-$convex stochastic processes. And, finally, if we choose $m=1, h_1(t)=t^s$ and $h_2(t)=(1-t)^s$ for $t\in[0,1]$ and some fixed $s\in(0,1]$ it is attained the inequality* (19) *in Theorem 3.9 for $s-$convex stochastic process in the second sense.*

Theorem 3.13. *Let $h_1,h_2:[0,1]\to\mathbb{R}$ be a non negative functions, $m\in(0,1]$ and $X: I\times\Omega\to\mathbb{R}$ be a mean square continuous and mean square integrable stochastic process. Let $p,q,k>0$, if $|X|^r$ is $(m,h_1,h_2)-$convex on $[a,b]$ for $r>1$, where $a,b\in I$ and $a<b$, then the following inequality holds almost everywhere*

$$\int_a^b (u-a)^{p/k}(b-u)^{q/k} X(u,\cdot)du \tag{26}$$

$$\leq k^{1/j}(b-a)^{\frac{p+q}{k}+1}\left(B_k(jp+k,jq+k)\right)^{1/j}\left(|X(a,\cdot)|^r I(h_1)+|X(b,\cdot)|^r mI(h_2)\right)^{1/r},$$

where

$$I(h_1) = \int_0^1 (1-t)^{p/k}\, t^{q/k} h_1(t)dt,$$

$$I(h_2) = \int_0^1 (1-t)^{p/k}\, t^{q/k} h_2(t)dt$$

and $1/j + 1/r = 1$.

Proof. From Lemma 3.1 and using the Hölder inequality we have

$$\begin{aligned} &\int_a^b (u-a)^{p/k}\, (b-u)^{q/k}\, X(u,\cdot)du \\ &\qquad \leq (b-a)^{\frac{p+q}{k}+1} \int_0^1 (1-t)^{p/k}\, t^{q/k}\, |X\,(ta+(1-t)b,\cdot)|\, dt \\ &\qquad \leq (b-a)^{\frac{p+q}{k}+1} \left(\int_0^1 (1-t)^{jp/k}\, t^{jq/k} dt \right)^{1/j} \left(\int_0^1 |X\,(ta+(1-t)b,\cdot)|^l\, dt \right)^{1/l}. \end{aligned} \tag{27}$$

Since $|X|^l$ is $(m, h_1, h_2)-$convex stochastic process then

$$\begin{aligned} &\int_0^1 |X\,(ta+(1-t)b,\cdot)|^l\, dt \\ &\qquad \leq \quad |X(a,\cdot)|^l \int_0^1 h_1(t)dt + |X\,(b,\cdot)|^l\, m \int_0^1 h_2(t)dt, \end{aligned} \tag{28}$$

and using the definition of the $k-$Beta function we get

$$\int_0^1 (1-t)^{jp/k}\, t^{jq/k} dt = kB_k(jp+k, jq+k). \tag{29}$$

So, replacing (28) and (29) in (27) it is attained the desired inequality (26).
The proof is complete. ■

Remark 3.14. *If in Theorem 3.13 we choose* $m = 1, h_1(t) = t$ *and* $h_2(t) = 1 - t$ *for* $t \in [0, 1]$ *then we obtain the inequality* (6) *in Theorem 3.3 for convex stochastic processes. Similarly, if we choose* $m = 1, h_1(t) = 1$ *and* $h_2(t) = 1$ *for* $t \in [0, 1]$ *then we get the inequality* (13) *in Theorem 3.6 for* $P-$*convex stochastic processes. And, finally, if we choose* $m = 1, h_1(t) = t^s$ *and* $h_2(t) = (1-t)^s$ *for* $t \in [0, 1]$ *and some fixed* $s \in (0, 1]$ *it is attained the inequality* (19) *in Theorem 3.9 for* $s-$*convex stochastic process in the second sense.*

Theorem 3.15. *Let* $h_1, h_2 : [0, 1] \to \mathbb{R}$ *be a non negative functions,* $m \in (0, 1]$ *and* $X : I \times \Omega \to \mathbb{R}$ *be a mean square continuous and mean square integrable stochastic process. Let* $p, q, k > 0$*, if* $|X|^r$ *is* $(m, h_1, h_2)-$*convex on* $[a, b]$ *for* $r > 1$*, where* $a, b \in I$ *and* $a < b$*, then the following inequality holds almost everywhere*

$$\int_a^b (u-a)^{p/k}\, (b-u)^{q/k}\, X(u,\cdot)du \tag{30}$$

$$\leq k\,(b-a)^{\frac{p+q}{k}+1}\left(\frac{pq}{(p+q)_{2,k}}B_k(p,q)\right)^{1-1/r}\left(|X(a,\cdot)|^r\,I(h_1)+|X(b,\cdot)|^r\,mI(h_2)\right)^{1/r},$$

where

$$I(h_1)=\int_0^1(1-t)^{p/k}\,t^{q/k}h_1(t)dt,$$
$$I(h_2)=\int_0^1(1-t)^{p/k}\,t^{q/k}h_2(t)dt.$$

Proof. From Lemma 3.1 and using the power mean inequality for $l>1$ we have

$$\begin{aligned}&\int_a^b(u-a)^{p/k}\,(b-u)^{q/k}\,X(u,\cdot)du \\ &\quad\leq (b-a)^{\frac{p+q}{k}+1}\int_0^1(1-t)^{p/k}\,t^{q/k}\,|X\,(ta+(1-t)b,\cdot)|\,dt \\ &\quad\leq (b-a)^{\frac{p+q}{k}+1}\left(\int_0^1(1-t)^{p/k}\,t^{q/k}dt\right)^{1-1/r}\times \\ &\qquad\qquad\left(\int_0^1(1-t)^{p/k}\,t^{q/k}\,|X\,(ta+(1-t)b,\cdot)|^r\,dt\right)^{1/r}.\end{aligned}\tag{31}$$

Since $|X|^r$ is $(m,h_1,h_2)-$convex and using the definition of the $k-$Beta function we get

$$\begin{aligned}&\int_0^1(1-t)^{p/k}\,t^{q/k}\,|X\,(ta+(1-t)b,\cdot)|^r\,dt \\ &\quad\leq\int_0^1(1-t)^{p/k}\,t^{q/k}\,(h_1(t)\,|X(a,\cdot)|^r+mh_2(t)\,|X(b,\cdot)|^r)\,dt \\ &\leq |X(a,\cdot)|^r\,I(h_1)+|X(b,\cdot)|^r\,mI(h_2),\end{aligned}$$

where

$$I(h_1)=\int_0^1(1-t)^{p/k}\,t^{q/k}h_1(t)dt,$$
$$I(h_2)=\int_0^1(1-t)^{p/k}\,t^{q/k}h_2(t)dt.$$

With this last result and again using the definition of the $k-$Beta function in the inequality (31) we obtain

$$\begin{aligned}&\int_a^b(u-a)^{p/k}\,(b-u)^{q/k}\,X(u,\cdot)du \\ &\quad\leq k\,(b-a)^{\frac{p+q}{k}+1}\left(\frac{pq}{(p+q)_{2,k}}B_k(p,q)\right)^{1-1/r}\left(|X(a,\cdot)|^r\,I(h_1)+|X(b,\cdot)|^r\,mI(h_2)\right)^{1/r}.\end{aligned}$$

The proof is complete. ■

Remark 3.16. *If in Theorem 3.13 we choose $m=1,h_1(t)=t$ and $h_2(t)=1-t$ for $t\in[0,1]$ then we obtain the inequality* (10) *in Theorem 3.4 for convex stochastic processes. Similarly, if we choose $m=1,h_1(t)=1$ and $h_2(t)=1$ for $t\in[0,1]$ then we get the inequality* (17) *in Theorem 3.7 for $P-$convex stochastic processes. And, finally, if we choose $m=1,h_1(t)=t^s$ and $h_2(t)=(1-t)^s$ for $t\in[0,1]$ and some fixed $s\in(0,1]$ it is attained the inequality* (23) *in Theorem 3.10 for $s-$convex stochastic process in the second sense.*

4. Conclusions

In the present article some integral inequalities involving the $k-$Beta function and stochastic processes whose absolute values posses the convexity, $P-$convexity, $s-$convexity or $(m,h_1,h_2)-$convexity property, including $r-$th powers of them, were established. Also, it is presented some consequences that derive from the theorems and that affirm the character of generalization that is attributed to the $(m,h_1,h_2)-$convex stochastic processes. Using this fact it is possible to find integral inequalities similar to those found in this work using other types of generalized inequalities such as: Godunova-Levin convexity, $(s,m)-$convexity, $MT-$convexity and others.

The author expect that this work will serve as stimulus for other research in this area.

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