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Hermite - Hadamard type inequalities for $(m, h_1, h_2)-$convex stochastic processes using Katugampola fractional integral.

Desigualdades del tipo Hermite-Hadamard para procesos estocásticos $(m, h_1, h_2)-$convexos usando la integral fraccional de Katugampola.

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Abstract

In this article some inequalities of the Hermite-Hadamard type are presented for $(m, h_1, h_2)-$convex stochastic processes using the fractional integral of Katugampola, and from these results specific cases are deduced for other stochastic processes with generalized convexity properties using the Riemann-Liouville fractional integral and the Riemann integral.

Keywords: Hermite-Hadamard Inequalities, $(m, h_1, h_2)-$convex stochastic processes, Katugampola Fractional integral
2015 MSC: 35A23, 60E15, 26A33

Resumen

En este artículo se presntan algunas desigualdades del tipo Hermite-Hadamard para procesos estocásticos $(m, h_1, h_2)-$convexos usando la integral fraccional de Katugampola, y a partir de estos resultados se deducen casos específicos para otros procesos estocásticos con propiedades de convexidad generalizada usando la integral fraccional de Riemann-Liouville y la clásica integral de Riemann.

Palabras claves: Desigualdad de Hermite-Hadamard, Procesos estocásticos $(m, h_1, h_2)-$convexos, Integral fraccional de Katugampola
2015 MSC: 35A23, 60E15, 26A33

1. Introduction

The study of convex functions has been of interest for mathematical analysis based on the properties that are deduced from this concept. Due to generalization requirements of the convexity concept, in order to obtain new applications, in the last years great efforts have been made in the study and investigation of this topic.

A function $f : I \to \mathbb{R}$ is said to be convex if for all $x, y \in I$ and $t \in [0, 1]$ the inequality

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$$

holds.

Numerous works of investigation have been realized extending results on inequalities for convex functions towards others much more generalized, using new concepts such as $E-$convexity ([45]), quasi-convexity ([37]), $s-$convexity ([3]), logarithmically convexity ([2]), and others.

A compendium about the history of the Hermite Hadamard inequality can be found in the work of D.S. Mitrinovic and I.B. Lackovic [30]. The formulation of this result is as follows:

(Hermite-Hadamard Inequality). Let $f : I \to \mathbb{R}$ be a convex function, and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

The inequality of Hermite Hadamard has become a very useful tool in the Theory of Probability and Optimization (See [26]).

The study on convex stochastic processes began in 1974 when B. Nagy in [31], applied a characterization of measurable stochastic processes to solving a generalization of the (additive) Cauchy functional equation. In 1980 Nikodem [32] considered convex stochastic processes. In 1992 and 1995 Skowronski [40, 41] obtained some further results on convex stochastic processes which generalize some known properties.

In the year 2014, E. Set et. al. in [36] investigated Hermite-Hadamard type inequalities for stochastic processes in the second sense. For other results related to stochastic processes see [5],[12],[28],[38], [39], where further references are given.

Also, Fractional calculus [15, 29] was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

In 2011, U. Katugampola presented a new fractional integral operator in [20] which generalizes the Riemann-Liouville and the Hadamard integrals into a single form, and various researchers have made use of this result in the field of convexity, generalized convexity and others ([7, 9, 10, 43]).

Recently, several Hermite-Hadamard type inequalities [18, 19, 27, 44] associated with fractional integrals have been investigated. Here, it is established several generalized Hermite-Hadamard type integral inequalities for Stochastic processes using Katugampola fractional integral operator.

2. Preliminaries

The following notions can be found in some text books and articles. The reader can be see [24, 25, 28, 40, 41].

Let $(\Omega, \mathcal{A}, \mu)$ be an arbitrary probability space. A function $X : \Omega \to \mathbb{R}$ is called a random variable if it is $\mathcal{A}$-measurable. Let $I \subset \mathbb{R}$ be time. A function $X : I \times \Omega \to \mathbb{R}$ is called stochastic process, if for all $u \in I$ the function $X(u, \cdot)$ is a random variable.

A stochastic process $X : I \times \Omega \to \mathbb{R}$ is called continuous in probability in the interval I if for all $t_0 \in I$ it is had that

$$\mu - \lim_{t \to t_0} X(t, \cdot) = X(t_0, \cdot),$$

where $\mu - \text{lím}$ denotes the limit in probability, and it is called mean-square continuous in the interval I if for all $t_0 \in I$

$$\mu - \lim_{t \to t_0} \mathbb{E}(X(t, \cdot) - X(t_0, \cdot)) = 0,$$

where $\mathbb{E}(X(t, \cdot))$ denote the expectation value of the random variable $X(t, \cdot)$.

Also, the monotony property it is attained. A stochastic process is called increasing (decreasing) if for all $u, v \in I$ such that $t < s$,

$$X(u, \cdot) \leq X(v, \cdot), \qquad (X(u, \cdot) \geq X(v, \cdot)) \quad (a.e.)$$

respectively, and is called monotonic if it's increasing or decreasing.

About differentiability it is said that a stochastic processes is differentiable at a point $t \in I$ if there is a random variable $X'(t, \cdot)$ such that

$$X'(t, \cdot) = \mu - \lim_{t \to t_0} \frac{X(t, \cdot) - X(t_0, \cdot)}{t - t_0}.$$

Let $[a, b] \subset I, a = t_0 < t_1 < ... < t_n = b$ be a partition of $[a, b]$ and $\theta_k \in [t_{k-1}, t_k]$ for $k = 1, 2, ..., n$. Let X be a stochastic process such that $\mathbb{E}(X(u, \cdot)^2) < \infty$. A random variable $Y : \Omega \to \mathbb{R}$ is called mean-square integral of the process $X(t, \cdot)$ on $[a, b]$ if the following identity holds:

$$\lim_{n \to \infty} E[X(\theta_k, \cdot)(t_k - t_{k-1}) - Y(\cdot)]^2 = 0$$

Then

$$\int_a^b X(t, \cdot) dt = Y(\cdot) \qquad (a.e.).$$

The book of K. Sobczyk [42] contains basic properties of mean-square integral.

Important theorems as the mean value theorem for mean square derivatives and integrals for stochastic processes have been proved in the work of J.C. Cortéz et. al. The reader can find these results in [8, Lemma 3.1,Theorem 3.2].

In 1980 K. Nikodem introduced the following definition [32].

Definition 2.1. *Set $(\Omega, \mathcal{A}, P)$ be a probability space and $I \subset \mathbb{R}$ be an interval. The stochastic process $X : I \times \Omega \to \mathbb{R}$ is said to be convex stochastic process if*

$$X(\lambda u + (1 - \lambda)v, \cdot) \leq \lambda X(u, \cdot) + (1 - \lambda)X(v, \cdot) \tag{1}$$

holds almost everywhere for all $u, v \in I$ and $\lambda \in [0, 1]$.

Using Definition 2.1, D. Kotrys presented, in 2012, the Hermite-Hadamard integral inequality version for Stochastic Processes [24].

Theorem 2.2. *If $X : I \times \Omega \to \mathbb{R}$ is convex and mean square continuous in the interval $T \times \Omega$, then for any $u, v \in T$, the inequality*

$$X\left(\frac{u + v}{2}, \cdot\right) \leq \frac{1}{u - v} \int_u^v X(t, \cdot) dt \leq \frac{X(u, \cdot) + X(v, \cdot)}{2}$$

holds almost everywhere.

There is also a generalization of the concept of convexity associated with stochastic processes. In [36] we find the following definition.

Definition 2.3. *Let $0 < s < 1$. A stochastic processes $X : I \times \Omega \to \mathbb{R}$ is said to be $s-$convex stochastic processes in the second sense if*

$$X(ta + (1-t)b, \cdot) \leq t^s X(a, \cdot) + (1-t)^s X(b, \cdot) \tag{2}$$

holds almost everywhere for any $a, b \in I$ and all $t \in [0, 1]$.

Also, Hernández Hernández J.E. and Gómez, J.F. introduced the definition of $P-$convex Stochastic processes and $(m, h_1, h_2)-$convex stochastic processes in [18] .

Definition 2.4. *A stochastic processes $X : I \times \Omega \to \mathbb{R}$ is said to be $P-$convex stochastic processes if*

$$X(ta + (1-t)b, \cdot) \leq X(a, \cdot) + X(b, \cdot) \tag{3}$$

holds almost everywhere for any $a, b \in I$ and all $t \in [0, 1]$.

Definition 2.5. *Let $m \in (0, 1]$ and $h_1, h_2 : [0, 1] \to \mathbb{R}^+$ be functions. A stochastic processes $X : I \times \Omega \to \mathbb{R}$ is said to be $(m, h_1, h_2)-$convex stochastic processes if*

$$X(ta + m(1-t)b, \cdot) \leq h_1(t)X(a, \cdot) + mh_2(t)X(b, \cdot) \tag{4}$$

holds almost everywhere for any $a, b \in I$ and all $t \in [0, 1]$.

Before establishing the main results, it will be given some necessary notions and mathematical preliminaries of fractional calculus theory which are used further in this paper. For more details, one can consult [15, 23, 29, 34].

Definition 2.6. *Let$f \in L_1([a, b])$. The Riemann-Liouville integrals J^{α}_{a+} and J^{α}_{b-} of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt$$

and

$$J^{\alpha}_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt$$

respectively, where $\Gamma(\alpha)$ is the Euler's Gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt.$$

Note that $J^0_{a+} f(x) = J^0_{b-} f(x) = f(x)$.

Using the Riemann-Liouville fractional integral, Sarikaya et. al. [35], established the Hermite-Hadamard inequalities version.

Theorem 2.7. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $a < b$ and $f \in L_1([a, b])$. If f is a convex function on $[a, b]$ then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left(J^{\alpha}_{a+} f(b) + J^{\alpha}_{b-} f(a)\right) \leq \frac{f(a) + f(b)}{2}$$

with $\alpha > 0$.

For stochastic processes, Agahi H. and Babakhani A. established this inequality type in [1].

Theorem 2.8. *Let $X : I \times \Omega \to \mathbb{R}$ be a Jensen-convex stochastic process that is meansquare continuous in the interval I . Then for any $u, v \in I$ with $u < v$, the following Hermite–Hadamard inequality*

$$X\left(\frac{u+v}{2}, \cdot\right) \leq \frac{\Gamma(\alpha+1)}{2(v-u)^{\alpha}}\left(J^{\alpha}_{a+}X(b,\cdot) + J^{\alpha}_{b-}X(a,\cdot)\right) \leq \frac{X(u,\cdot)+X(v,\cdot)}{2} \quad (a.e)$$

holds, where $\alpha > 0$.

Also, J. Hadamard in 1892 introduced the following fractional integral ope-rator ([17]).

Definition 2.9. *Let $\alpha > 0$ with $n-1 < \alpha < n$, $n \in \mathbb{N}$, and $a < x < b$. The left- and right-side Hadamard fractional integrals of order $\alpha > 0$ of a function f, are given by*

$$H^{\alpha}_{a+}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1}\frac{f(t)}{t}dt$$

and

$$H^{\alpha}_{b-}f(t) = \frac{1}{\Gamma(\alpha)}\int_x^b \left(\ln\frac{x}{t}\right)^{\alpha-1}\frac{f(t)}{t}dt$$

respectively.

As it was mentioned in the introductory section, Katugampola introduced a new fractional integral that generalizes the Riemann-Liouville and Hadamard fractional integrals into a single form (see [20, 21, 22]).

In the following will denote the space $X^p_c(a,b)$, $(c \in \mathbb{R}, 1 \leq p \leq \infty)$ of those complex valued Lebesgue measurable functions f on $[a,b]$ for which $\|f\|_{X^p_c} < \infty$ where

$$\|f\|_{X^p_c} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t}\right)^{1/p}.$$

Katugampola in [21] established the following definition and property.

Definition 2.10. *Let $[a,b] \subset \mathbb{R}$ be a finite interval. The left side and right side Katugampola fractional integral of order $\alpha > 0$ of $f \in X^p_c(a,b)$ are defined by*

$${}^{\rho}I^{\alpha}_{a+}f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x \frac{t^{\rho-1}}{(x^{\rho}-t^{\rho})^{1-\alpha}}f(t)dt$$

and

$${}^{\rho}I^{\alpha}_{b-}f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_x^b \frac{t^{\rho-1}}{(t^{\rho}-x^{\rho})^{1-\alpha}}f(t)dt,$$

with $a < x < b$ and $\rho > 0$, if the integrals exist.

Theorem 2.11. *Let $\alpha > 0$ and $\rho > 0$. For $x > a$*

$$\lim_{\rho\to 1} {}^{\rho}I^{\alpha}_{a+}f(x) = J^{\alpha}_{a+}f(x)$$

and

$$\lim_{\rho\to 0^+} {}^{\rho}I^{\alpha}_{a+}f(x) = H^{\alpha}_{a+}f(x).$$

Similar results also hold for the right-sided operators.

For this kind of fractional integral operator is proved the following Theorem which establish the Hermite-Hadamard inequality [19].

Theorem 2.12. *Let $\alpha > 0$ and $\rho > 0$. Let $X : [a^\rho, b^\rho] \times \Omega \to \mathbb{R}$ be a positive stochastic process with $0 \leq a < b$ and $X(t,\cdot) \in X_c^p(a^\rho, b^\rho)$. If $X(t,\cdot)$ is convex, the following inequality holds almost everywhere:*

$$X\left(\frac{a^\rho + b^\rho}{2}, \cdot\right) \leq \frac{\Gamma(\alpha+1)}{2\rho^{-\alpha}(b^\rho - a^\rho)^\alpha} \left({}^\rho I^\alpha_{b^\rho -} X(a^\rho, \cdot) + {}^\rho I^\alpha_{a^\rho +} X(b^\rho, \cdot)\right) \leq \frac{X(a^\rho, \cdot) + X(b^\rho, \cdot)}{2\rho\alpha}$$

In the same reference is proved the following Lemma.

Lemma 2.13. *Let $X : [a^\rho, b^\rho] \times \Omega \to \mathbb{R}$ be a mean square differentiable stochastic process; then the following equality holds:*

$$\frac{X(a^\rho, \cdot) + X(b^\rho, \cdot)}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha} \left({}^\rho I^\alpha_{b^\rho -} X(a^\rho, \cdot) + {}^\rho I^\alpha_{a^\rho +} X(b^\rho, \cdot)\right)$$
$$= \frac{\rho(b^\rho - a^\rho)}{2} \int_0^1 \left[(1-t^\rho)^\alpha - t^{\rho\alpha}\right] t^{\rho-1} X'(t^\rho a^\rho + (1-t^\rho) b^\rho, \cdot)\, dt.$$

The purpose of this paper is to derive some inequalities of type Hermite-Hadamard for $(m, h_1, h_2)-$convex stochastic processes using the Katugampola fractional integrals.

3. Main Results

Theorem 3.1. *Let $\alpha > 0$ and $\rho > 0$. Let $X : [a^\rho, b^\rho] \times \Omega \to \mathbb{R}$ be a positive stochastic process with $0 \leq a < b$ and $X(t,\cdot) \in X_c^p(a^\rho, b^\rho)$. If $X(t,\cdot)$ is $(m, h_1, h_2)-$convex the the following inequalities holds almost everywhere*

$$X\left(\frac{a^\rho + b^\rho}{2}, \cdot\right) \leq \frac{\rho^\alpha \Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha} \left(h_1(1/2)\, {}^\rho I^\alpha_{b-} X(a^\rho, \cdot) + h_2(1/2)\, {}^\rho I^\alpha_{a+} X\left(\frac{b^\rho}{m}, \cdot\right)\right) \tag{5}$$

and

$$\frac{\Gamma(\alpha)}{\rho^{1-\alpha}(b^\rho - a^\rho)^\alpha} \left({}^\rho I^\alpha_{b^\rho -} X(a^\rho, \cdot) + {}^\rho I^\alpha_{a^\rho +} X(b^\rho, \cdot)\right)$$
$$\leq \left(X(a^\rho, \cdot) + X(b^\rho, \cdot)\right) \mathtt{I}(h_1) + \left(X\left(\frac{a^\rho}{m}, \cdot\right) + X\left(\frac{b^\rho}{m}, \cdot\right)\right) \mathtt{I}(h_2), \tag{6}$$

where

$$\mathtt{I}(h_1) = \int_0^1 t^{\alpha\rho - 1} h_1(t^\rho) dt \ \ and \ \ \mathtt{I}(h_2) = \int_0^1 t^{\alpha\rho-1} h_2(t^\rho) dt.$$

Proof. Let $t \in [0,1]$, and $u, v \in [a,b]$ defined by

$$u^\rho = t^\rho a^\rho + (1-t^\rho) b^\alpha \ \text{ and } \ v^\rho = (1-t^\rho) a^\rho + t^\rho b^\rho. \tag{7}$$

then,

$$X\left(\frac{a^\rho + b^\rho}{2}, \cdot\right) = X\left(\frac{u^\rho + v^\rho}{2}, \cdot\right) = X\left(\frac{t^\rho a^\rho + (1-t^\rho) b^\alpha + (1-t^\rho) a^\rho + t^\rho b^\rho}{2}\right).$$

Using the $(m, h_1, h_2)-$convexity of X,

$$X\left(\frac{a^\rho + b^\rho}{2}, \cdot\right)$$

$$\leq h_1\left(\frac{1}{2}\right)X\left(t^\rho a^\rho+(1-t^\rho)\,b^\alpha,\cdot\right)+mh_2\left(\frac{1}{2}\right)X\left((1-t^\rho)\,\frac{a^\rho}{m}+t^\rho\frac{b^\rho}{m},\cdot\right). \tag{8}$$

Multiplying both side of (8) by $t^{\alpha\rho-1}, (\alpha,\rho>0)$ and integrating over $t\in[0,1]$, it is obtained that

$$\begin{aligned}&\frac{1}{\alpha\rho}X\left(\frac{a^\rho+b^\rho}{2},\cdot\right)\\ &\leq h_1\left(\frac{1}{2}\right)\int_0^1 t^{\alpha\rho-1}X\left(t^\rho a^\rho+(1-t^\rho)\,b^\alpha,\cdot\right)dt\\ &\qquad+mh_2\left(\frac{1}{2}\right)\int_0^1 t^{\alpha\rho-1}X\left((1-t^\rho)\,\frac{a^\rho}{m}+t^\rho\frac{b^\rho}{m},\cdot\right)dt.\end{aligned} \tag{9}$$

Now, from (7) and the Definition 2.10, it is obtained that

$$\begin{aligned}\int_0^1 t^{\alpha\rho-1}X\left(t^\rho a^\rho+(1-t^\rho)b^\alpha,\cdot\right)dt &= \frac{1}{(b^\rho-a^\rho)^\alpha}\int_a^b\frac{u^{\rho-1}}{(u^\rho-b^\rho)^{1-\alpha}}X\left(u^\rho,\cdot\right)du\\ &= \frac{\Gamma(\alpha)}{\rho^{1-\alpha}\,(b^\rho-a^\rho)^\alpha}\,{}^\rho I^\alpha_{b-}X\left(a^\rho,\cdot\right)\end{aligned} \tag{10}$$

and

$$\int_0^1 t^{\alpha\rho-1}X\left((1-t^\rho)\,\frac{a^\rho}{m}+t^\rho\frac{b^\rho}{m},\cdot\right)dt=\frac{\Gamma(\alpha)}{\rho^{1-\alpha}\,(b^\rho-a^\rho)^\alpha}\,{}^\rho I^\alpha_{a+}X\left(\frac{b^\rho}{m},\cdot\right). \tag{11}$$

Replacing (10) and (11) in (9), it is attained (5)

$$X\left(\frac{a^\rho+b^\rho}{2},\cdot\right)\leq\frac{\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left(h_1(1/2)\,{}^\rho I^\alpha_{b-}X\left(a^\rho,\cdot\right)+h_2(1/2)\,{}^\rho I^\alpha_{a+}X\left(\frac{b^\rho}{m},\cdot\right)\right).$$

In order to obtain (6) , it is used the $(m,h_1,h_2)-$convexity property of the stochastic process X

$$X\left(t^\rho a^\rho+(1-t^\rho)b^\alpha,\cdot\right)\leq h_1(t^\rho)X\left(a^\rho,\cdot\right)+mh_2(t^\rho)X\left(\frac{b^\rho}{m},\cdot\right)$$

$$X\left((1-t^\rho)\,a^\rho+t^\rho b^\rho,\cdot\right)\leq h_1(t^\rho)X\left(b^\rho,\cdot\right)+mh_2(t^\rho)X\left(\frac{a^\rho}{m},\cdot\right),$$

adding these inequalities it is obtained

$$\begin{aligned}&X\left(t^\rho a^\rho+(1-t^\rho)b^\alpha,\cdot\right)+X\left((1-t^\rho)\,a^\rho+t^\rho b^\rho,\cdot\right)\\ &\leq\left(X\left(a^\rho,\cdot\right)+X\left(b^\rho,\cdot\right)\right)h_1(t^\rho)+m\left(X\left(\frac{a^\rho}{m},\cdot\right)+X\left(\frac{b^\rho}{m},\cdot\right)\right)h_2(t^\rho).\end{aligned} \tag{12}$$

Multiplying both side of (12) by $t^{\alpha\rho-1}, (\alpha,\rho>0)$ and integrating over $t\in[0,1]$, it is attained that

$$\begin{aligned}&\frac{\Gamma(\alpha)}{\rho^{1-\alpha}\,(b^\rho-a^\rho)^\alpha}\left({}^\rho I^\alpha_{b^\rho-}X\left(a^\rho,\cdot\right)+\,{}^\rho I^\alpha_{a^\rho+}X\left(b^\rho,\cdot\right)\right)\\ &\leq\left(X\left(a^\rho,\cdot\right)+X\left(b^\rho,\cdot\right)\right)\mathtt{I}(h_1)+\left(X\left(\frac{a^\rho}{m},\cdot\right)+X\left(\frac{b^\rho}{m},\cdot\right)\right)\mathtt{I}(h_2),\end{aligned}$$

where

$$\mathtt{I}(h_1)=\int_0^1 t^{\alpha\rho-1}h_1(t^\rho)dt\ \text{ and }\ \mathtt{I}(h_2)=\int_0^1 t^{\alpha\rho-1}h_2(t^\rho)dt.$$

The proof is complete. ■

Remark 3.2. *Letting $m = 1$ and $h_1(t) = t, h_2(t) = 1 - t$ for $t \in [0, 1]$ in Theorem 3.1, it is had that*

$$\mathtt{I}(h_1) = \int_0^1 t^{\alpha\rho-1} t^\rho dt = \int_0^1 t^{\rho(\alpha+1)-1} dt = \frac{1}{\rho(\alpha+1)},$$

$$\mathtt{I}(h_2) = \int_0^1 t^{\alpha\rho-1}(1 - t^\rho) dt = \frac{1}{\rho\alpha} - \frac{1}{\rho(\alpha+1)}$$

and

$$h_1(1/2) = h_2(1/2) = \frac{1}{2}$$

so, for convex stochastic processes it is had that

$$X\left(\frac{a^\rho + b^\rho}{2}, \cdot\right) \leq \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha} \left({}^\rho I^\alpha_{b-} X(a^\rho, \cdot) + {}^\rho I^\alpha_{a+} X(b^\rho, \cdot)\right) \leq \frac{X(a^\rho, \cdot) + X(b^\rho, \cdot)}{2\rho\alpha}$$

almost everywhere, making coincidence with Theorem 3.1 in [19]; using the Theorem 2.11 it is obtained the Hermite-Hadamard inequality version for convex stochastic processes and the Riemann Liouville fractional integral

$$X\left(\frac{a+b}{2}, \cdot\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left(J^\alpha_{b-} X(a, \cdot) + J^\alpha_{a-} X(b, \cdot)\right) \leq \frac{X(a, \cdot) + X(b, \cdot)}{2\alpha}, \tag{13}$$

almost everywhere, making coincidence with the result proved by H. Aghahi and A. Babakhani in [1]. Letting $\alpha = 1$ in (13) it is obtained the Hermite Hadamard inequality for the classical Riemann integral

$$X\left(\frac{a+b}{2}, \cdot\right) \leq \frac{1}{(b-a)} \int_a^b X(t, \cdot)\, dt \leq \frac{X(a, \cdot) + X(b, \cdot)}{2}$$

almost evrywhere, making coincidence with the result proved by Kotrys in [24].

Theorem 3.3. *Let $\alpha > 0$ and $\rho > 0$. Let $X : [a^\rho, b^\rho] \times \Omega \to \mathbb{R}$ be a mean square differentiable stochastic process with $0 \leq a < b$ and $X(t, \cdot) \in X^p_c(a^\rho, b^\rho)$. If $|X'(t, \cdot)|$ is $(m, h_1, h_2)-$convex then the following inequality holds almost everywhere*

$$\begin{aligned}&\left|\frac{X(a^\rho, \cdot) + X(b^\rho, \cdot)}{2\alpha\rho} - \frac{\Gamma(\alpha)}{2\rho^{1-\alpha}(b^\rho - a^\rho)^\alpha} \left({}^\rho I^\alpha_{b^\rho-} X(a^\rho, \cdot) + {}^\rho I^\alpha_{a^\rho+} X(b^\rho, \cdot)\right)\right| \\ &\quad \leq \frac{(b^\rho - a^\rho)}{2\alpha} \left(\left(\left|X'(a^\rho \cdot)\right| + \left|X'(b^\rho, \cdot)\right|\right) \mathtt{I}(h_1) + \left(\left|X'\left(\frac{a^\rho}{m}, \cdot\right)\right| + \left|X'\left(\frac{b^\rho}{m}, \cdot\right)\right|\right) \mathtt{I}(h_2)\right),\end{aligned} \tag{14}$$

where

$$\mathtt{I}(h_1) = \int_0^1 t^{\rho(\alpha+1)-1} h_1(t^\rho) dt \;\; and \;\; \mathtt{I}(h_2) = \int_0^1 t^{\rho(\alpha+1)-1} h_2(t^\rho) dt.$$

Proof. From the Definition 2.10 and a suitable change of variables we get

$$\frac{\Gamma(\alpha)}{\rho^{1-\alpha}(b^\rho - a^\rho)^\alpha} \left({}^\rho I^\alpha_{b^\rho-} X(a^\rho, \cdot) + {}^\rho I^\alpha_{a^\rho+} X(b^\rho, \cdot)\right)$$

$$= \int_0^1 t^{\alpha\rho-1} X\left(t^\rho a^\rho + (1-t^\rho) b^\alpha, \cdot\right) dt + \int_0^1 t^{\alpha\rho-1} X\left((1-t^\rho) a^\rho + t^\rho b^\rho, \cdot\right) dt \tag{15}$$

Integrating by parts each of the integrals we have

$$\begin{aligned}
&\int_0^1 t^{\alpha\rho-1} X\left(t^\rho a^\rho + (1-t^\rho) b^\alpha, \cdot\right) dt \\
&= \left.\frac{t^{\alpha\rho} X\left(t^\rho a^\rho + (1-t^\rho) b^\alpha, \cdot\right)}{\alpha\rho}\right|_0^1 - \frac{(a^\rho - b^\rho)}{\alpha} \int_0^1 t^{\alpha\rho+\rho-1} X'\left(t^\rho a^\rho + (1-t^\rho) b^\alpha, \cdot\right) dt \\
&= \frac{X(a^\rho, \cdot)}{\alpha\rho} - \frac{(a^\rho - b^\rho)}{\alpha} \int_0^1 t^{\rho(\alpha+1)-1} X'\left(t^\rho a^\rho + (1-t^\rho) b^\alpha, \cdot\right) dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 t^{\alpha\rho-1} X\left((1-t^\rho) a^\rho + t^\rho b^\rho, \cdot\right) dt \\
&= \left.\frac{t^{\alpha\rho} X\left((1-t^\rho) a^\rho + t^\rho b^\rho, \cdot\right)}{\alpha\rho}\right|_0^1 - \frac{(b^\rho - a^\rho)}{\alpha} \int_0^1 t^{\alpha\rho+\rho-1} X'\left((1-t^\rho) a^\rho + t^\rho b^\rho, \cdot\right) dt \\
&= \frac{X(b^\rho, \cdot)}{\alpha\rho} - \frac{(b^\rho - a^\rho)}{\alpha} \int_0^1 t^{\rho(\alpha+1)-1} X'\left((1-t^\rho) a^\rho + t^\rho b^\rho, \cdot\right) dt.
\end{aligned}$$

So

$$\begin{aligned}
\frac{\Gamma(\alpha)}{\rho^{1-\alpha}(b^\rho - a^\rho)^\alpha} \left({}^\rho I^\alpha_{b^\rho-} X(a^\rho, \cdot) + {}^\rho I^\alpha_{a^\rho+} X(b^\rho, \cdot)\right) &= \frac{X(a^\rho, \cdot) + X(b^\rho, \cdot)}{\alpha\rho} - \frac{(b^\rho - a^\rho)}{\alpha} \times \\
&\left(\int_0^1 t^{\rho(\alpha+1)-1} \left(X'\left(t^\rho a^\rho + (1-t^\rho) b^\alpha, \cdot\right) - X'\left((1-t^\rho) a^\rho + t^\rho b^\rho, \cdot\right)\right) dt\right)
\end{aligned} \tag{16}$$

By means of the equality (16), the triangular inequality and the (m, h_1, h_2)–convexity of $|X'(t, \cdot)|$, it is obtained that

$$\begin{aligned}
&\left|\frac{X(a^\rho, \cdot) + X(b^\rho, \cdot)}{\alpha\rho} - \frac{\Gamma(\alpha)}{\rho^{1-\alpha}(b^\rho - a^\rho)^\alpha} \left({}^\rho I^\alpha_{b^\rho-} X(a^\rho, \cdot) + {}^\rho I^\alpha_{a^\rho+} X(b^\rho, \cdot)\right)\right| \\
&\leq \frac{(b^\rho - a^\rho)}{\alpha} \int_0^1 t^{\rho(\alpha+1)-1} \left|X'\left(t^\rho a^\rho + (1-t^\rho) b^\alpha, \cdot\right) - X'\left((1-t^\rho) a^\rho + t^\rho b^\rho, \cdot\right)\right| dt \\
&\leq \frac{(b^\rho - a^\rho)}{\alpha} \left(\int_0^1 t^{\rho(\alpha+1)-1} \left|X'\left(t^\rho a^\rho + (1-t^\rho) b^\alpha, \cdot\right)\right| dt + \int_0^1 t^{\rho(\alpha+1)-1} \left|X'\left((1-t^\rho) a^\rho + t^\rho b^\rho, \cdot\right)\right| dt\right) \\
&\leq \frac{(b^\rho - a^\rho)}{\alpha} \left(\int_0^1 t^{\rho(\alpha+1)-1} \left(h_1(t^\rho) \left|X'(a^\rho \cdot)\right| + m h_2(t^\rho) \left|X'\left(\frac{b^\rho}{m} \cdot\right)\right|\right) dt\right. \\
&\qquad \left. + \int_0^1 t^{\rho(\alpha+1)-1} \left(m h_2(t^\rho) \left|X'\left(\frac{a^\rho}{m}, \cdot\right)\right| + h_1(t^\rho) \left|X'(b^\rho, \cdot)\right|\right) dt\right) \\
&= \frac{(b^\rho - a^\rho)}{\alpha} \left(\left(\left|X'(a^\rho \cdot)\right| + \left|X'(b^\rho, \cdot)\right|\right) \int_0^1 t^{\rho(\alpha+1)-1} h_1(t^\rho) dt\right.
\end{aligned}$$

$$+\left(\left|X'\left(\frac{a^\rho}{m},\cdot\right)\right|+\left|X'\left(\frac{b^\rho}{m},\cdot\right)\right|\right)\int_0^1 t^{\rho(\alpha+1)-1}h_2(t^\rho)dt\right),$$

and doing

$$\mathtt{I}(h_1)=\int_0^1 t^{\rho(\alpha+1)-1}h_1(t^\rho)dt \ \text{ and } \ \mathtt{I}(h_2)=\int_0^1 t^{\rho(\alpha+1)-1}h_2(t^\rho)dt.$$

it is attained the desired result.

The proof is complete. ■

Remark 3.4. *Letting $m=1$ and $h_1(t)=t, h_2(t)=1-t$ for $t\in[0,1]$ in Theorem 3.3, it is had that*

$$\mathtt{I}(h_1)=\int_0^1 t^{\rho(\alpha+1)-1}t^\rho dt=\int_0^1 t^{\rho(\alpha+2)-1}dt=\frac{1}{\rho(\alpha+2)}$$

and

$$\mathtt{I}(h_2)=\int_0^1 t^{\rho(\alpha+1)-1}(1-t^\rho)dt=\frac{1}{\rho(\alpha+1)}-\frac{1}{\rho(\alpha+2)}$$

so, for convex stochastic processes it is had that

$$\left|\frac{X(a^\rho,\cdot)+X(b^\rho,\cdot)}{2\alpha\rho}-\frac{\rho^{\alpha-1}\Gamma(\alpha)}{2(b^\rho-a^\rho)^\alpha}\left({}^\rho I^\alpha_{b^\rho-}X(a^\rho,\cdot)+{}^\rho I^\alpha_{a^\rho+}X(b^\rho,\cdot)\right)\right|\le\frac{(b^\rho-a^\rho)}{2\rho\alpha(\alpha+1)}\left(\left|X'(a^\rho,\cdot)\right|+\left|X'(b^\rho,\cdot)\right|\right).$$

Using the Theorem 2.11 it is obtained the Hermite-Hadamard inequality version for convex stochastic processes and the Riemann Liouville fractional integral

$$\left|\frac{X(a,\cdot)+X(b,\cdot)}{2\alpha}-\frac{\Gamma(\alpha)}{2(b-a)^\alpha}\left(J^\alpha_{b-}X(a,\cdot)+J^\alpha_{a+}X(b,\cdot)\right)\right|\le\frac{(b-a)}{2\alpha(\alpha+1)}\left(\left|X'(a,\cdot)\right|+\left|X'(b,\cdot)\right|\right).$$

and if $\alpha=1$ then

$$\left|\frac{X(a,\cdot)+X(b,\cdot)}{2}-\frac{1}{(b-a)}\int_a^b X(t,\cdot)dt\right|\le\frac{(b-a)}{4}\left(\left|X'(a,\cdot)\right|+\left|X'(b,\cdot)\right|\right).$$

Theorem 3.5. *Let $\alpha>0$ and $\rho>0$. Let $X:[a^\rho,b^\rho]\times\Omega\to\mathbb{R}$ be a mean square differentiable stochastic process with $0\le a<b$ and $X(t,\cdot)\in X^p_c(a^\rho,b^\rho)$. If $|X'(t,\cdot)|$ is $(m,h_1,h_2)-$convex then the following inequality holds almost everywhere*

$$\left|\frac{X(a^\rho,\cdot)+X(b^\rho,\cdot)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left({}^\rho I^\alpha_{b^\rho-}X(a^\rho,\cdot)+{}^\rho I^\alpha_{a^\rho+}X(b^\rho,\cdot)\right)\right|$$
$$\le\frac{b^\rho-a^\rho}{2}\left(\left|X'(a^\rho,\cdot)\right|S(h_1)+m\left|X(b^\rho/m,\cdot)\right|S(h_2)\right)$$

where

$$S(h_1)=\int_0^{1/2^{\frac{1}{\rho}}}\left((1-t^\rho)^\alpha-t^{\rho\alpha}\right)t^{\rho-1}h_1(t^\rho)dt+\int_{1/2^{\frac{1}{\rho}}}^1\left(t^{\rho\alpha}-(1-t^\rho)^\alpha\right)t^{\rho-1}h_1(t^\rho)dt$$

and

$$S(h_2)=\int_0^{1/2^{\frac{1}{\rho}}}\left((1-t^\rho)^\alpha-t^{\rho\alpha}\right)t^{\rho-1}h_2(t^\rho)dt+\int_{1/2^{\frac{1}{\rho}}}^1\left(t^{\rho\alpha}-(1-t^\rho)^\alpha\right)t^{\rho-1}h_2(t^\rho)dt$$

Proof. Using Lemma 2.13, the triangular inequality and the $(m, h_1, h_2)-$convexity of $|X|$ it is had that

$$\begin{aligned}
&\left|\frac{X(a^\rho,\cdot)+X(b^\rho,\cdot)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left({}^\rho I^\alpha_{b^\rho-}X(a^\rho,\cdot)+{}^\rho I^\alpha_{a^\rho+}X(b^\rho,\cdot)\right)\right|\\
&\quad\le\frac{\rho(b^\rho-a^\rho)}{2}\int_0^1\left|(1-t^\rho)^\alpha-t^{\rho\alpha}\right|t^{\rho-1}\left|X'(t^\rho a^\rho+(1-t^\rho)b^\rho,\cdot)\right|dt\\
&\quad\le\frac{\rho(b^\rho-a^\rho)}{2}\left(\int_0^{1/2^{\frac{1}{\rho}}}\left((1-t^\rho)^\alpha-t^{\rho\alpha}\right)t^{\rho-1}\left|X'(t^\rho a^\rho+(1-t^\rho)b^\rho,\cdot)\right|dt\right.\\
&\qquad\left.+\int_{1/2^{\frac{1}{\rho}}}^1\left(t^{\rho\alpha}-(1-t^\rho)^\alpha\right)t^{\rho-1}\left|X'(t^\rho a^\rho+(1-t^\rho)b^\rho,\cdot)\right|dt\right)\\
&\le\frac{\rho(b^\rho-a^\rho)}{2}\left(\int_0^{1/2^{\frac{1}{\rho}}}\left((1-t^\rho)^\alpha-t^{\rho\alpha}\right)t^{\rho-1}\left(h_1(t^\rho)\left|X'(a^\rho,\cdot)\right|+mh_2(t^\rho)\left|X(b^\rho/m,\cdot)\right|\right)dt\right.\\
&\qquad\left.+\int_{1/2^{\frac{1}{\rho}}}^1\left(t^{\rho\alpha}-(1-t^\rho)^\alpha\right)t^{\rho-1}\left(h_1(t^\rho)\left|X'(a^\rho,\cdot)\right|+mh_2(t^\rho)\left|X(b^\rho/m,\cdot)\right|\right)dt\right)\\
&\le\frac{\rho(b^\rho-a^\rho)}{2}\left(\left|X'(a^\rho,\cdot)\right|S(h_1)+m\left|X(b^\rho/m,\cdot)\right|S(h_2)\right)
\end{aligned}$$

where

$$S(h_1)=\int_0^{1/2^{\frac{1}{\rho}}}\left((1-t^\rho)^\alpha-t^{\rho\alpha}\right)t^{\rho-1}h_1(t^\rho)dt+\int_{1/2^{\frac{1}{\rho}}}^1\left(t^{\rho\alpha}-(1-t^\rho)^\alpha\right)t^{\rho-1}h_1(t^\rho)dt$$

and

$$S(h_2)=\int_0^{1/2^{\frac{1}{\rho}}}\left((1-t^\rho)^\alpha-t^{\rho\alpha}\right)t^{\rho-1}h_2(t^\rho)dt+\int_{1/2^{\frac{1}{\rho}}}^1\left(t^{\rho\alpha}-(1-t^\rho)^\alpha\right)t^{\rho-1}h_2(t^\rho)dt.$$

The proof is complete. ■

Remark 3.6. *Letting $m = 1$ and $h_1(t) = t, h_2(t) = 1 - t$ for $t \in [0, 1]$ in Theorem 3.5 it follows that*

$$\begin{aligned}
S(h_1)&=\int_0^{1/2^{\frac{1}{\rho}}}\left((1-t^\rho)^\alpha-t^{\rho\alpha}\right)t^{\rho-1}t^\rho dt+\int_{1/2^{\frac{1}{\rho}}}^1\left(t^{\rho\alpha}-(1-t^\rho)^\alpha\right)t^{\rho-1}t^\rho dt\\
&=\frac{1}{\rho(\alpha+1)}\left[1-\frac{1}{2^\alpha}\right]
\end{aligned}$$

and

$$\begin{aligned}
S(h_2)&=\int_0^{1/2^{\frac{1}{\rho}}}\left((1-t^\rho)^\alpha-t^{\rho\alpha}\right)t^{\rho-1}(1-t^\rho)dt+\int_{1/2^{\frac{1}{\rho}}}^1\left(t^{\rho\alpha}-(1-t^\rho)^\alpha\right)t^{\rho-1}(1-t^\rho)dt\\
&=\frac{1}{\rho(\alpha+1)}\left[1-\frac{1}{2^{\alpha+1}}\right]
\end{aligned}$$

and in consecuence it follows the Theorem 3.8 in [19] for convex stochastic processes, and in consequence the Remark 3.9 in the same reference.

4. Some Consequences

Corollary 4.1. *Let $\alpha > 0$ and $\rho > 0$. Let $X : [a^\rho, b^\rho]\times\Omega \to \mathbb{R}$ be a positive stochastic process with $0 \leq a < b$ and $X(t,\cdot) \in X_c^p(a^\rho, b^\rho)$. If $X(t,\cdot)$ is $s-$convex in the second sense the following inequalities holds almost everywhere*

$$X\left(\frac{a^\rho + b^\rho}{2}, \cdot\right) \leq \frac{\rho^\alpha\Gamma(\alpha+1)}{2^s(b^\rho - a^\rho)^\alpha}\left({}^\rho I_{b-}^\alpha X(a^\rho,\cdot) + {}^\rho I_{a+}^\alpha X(b^\rho,\cdot)\right) \leq \frac{X(a^\rho,\cdot) + X(b^\rho,\cdot)}{2^s}\left(\frac{1}{\alpha\rho + s} + B(\alpha\rho, s+1)\right)$$

Proof. Letting $m = 1$, $h_1(t) = t^s, h_2(t) = (1-t)^s$ for all $t \in [0,1]$ and $s \in (0,1]$ in Theorem 3.1 it is had that

$$\mathtt{I}(h_1) = \int_0^1 t^{\alpha\rho+s-1}dt = \frac{1}{\alpha\rho + s}, \mathtt{I}(h_2) = \int_0^1 t^{\alpha\rho-1}(1-t)^s dt = B(\alpha\rho, s+1),$$

and

$$h_1(1/2) = h_2(1/2) = \frac{1}{2^s}.$$

Therefore

$$X\left(\frac{a^\rho + b^\rho}{2}, \cdot\right) \leq \frac{\rho^\alpha\Gamma(\alpha+1)}{2^s(b^\rho - a^\rho)^\alpha}\left({}^\rho I_{b-}^\alpha X(a^\rho,\cdot) + {}^\rho I_{a+}^\alpha X(b^\rho,\cdot)\right) \leq \frac{X(a^\rho,\cdot) + X(b^\rho,\cdot)}{2^s}\left(\frac{1}{\alpha\rho + s} + B(\alpha\rho, s+1)\right)$$

The proof is complete. ■

Corollary 4.2. *Let $\alpha > 0$ and $\rho > 0$. Let $X : [a^\rho, b^\rho]\times\Omega \to \mathbb{R}$ be a positive stochastic process with $0 \leq a < b$ and $X(t,\cdot) \in X_c^p(a^\rho, b^\rho)$. If $X(t,\cdot)$ is $P-$convex the following inequalities holds almost everywhere*

$$X\left(\frac{a^\rho + b^\rho}{2}, \cdot\right) \leq \frac{\rho^\alpha\Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha}\left({}^\rho I_{b-}^\alpha X(a^\rho,\cdot) + {}^\rho I_{a+}^\alpha X(b^\rho,\cdot)\right) \leq \frac{X(a^\rho,\cdot) + X(b^\rho,\cdot)}{\alpha\rho}$$

Proof. Letting $m = 1$, $h_1(t) = h_2(t) = 1$ for all $t \in [0,1]$ in Theorem 3.1 it is had that

$$\mathtt{I}(h_1) = \mathtt{I}(h_2) = \int_0^1 t^{\alpha\rho-1}dt = \frac{1}{\alpha\rho}, h_1(1/2) = h_2(1/2) = 1.$$

Therefore

$$X\left(\frac{a^\rho + b^\rho}{2}, \cdot\right) \leq \frac{\rho^\alpha\Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha}\left({}^\rho I_{b-}^\alpha X(a^\rho,\cdot) + {}^\rho I_{a+}^\alpha X(b^\rho,\cdot)\right) \leq \frac{X(a^\rho,\cdot) + X(b^\rho,\cdot)}{\alpha\rho}$$

■

Corollary 4.3. *Let $\alpha > 0$ and $\rho > 0$. Let $X : [a^\rho, b^\rho] \times \Omega \to \mathbb{R}$ be a mean square differentiable stochastic process with $0 \leq a < b$ and $X(t,\cdot) \in X_c^p(a^\rho, b^\rho)$. If $|X'(t,\cdot)|$ is $s-$convex in the second sense then the following inequality holds almost everywhere*

$$\left|\frac{X(a^\rho,\cdot) + X(b^\rho,\cdot)}{2\alpha\rho} - \frac{\Gamma(\alpha)}{2\rho^{1-\alpha}(b^\rho - a^\rho)^\alpha}\left({}^\rho I_{b^\rho-}^\alpha X(a^\rho,\cdot) + {}^\rho I_{a^\rho+}^\alpha X(b^\rho,\cdot)\right)\right|$$
$$\leq \frac{(b^\rho - a^\rho)}{\rho\alpha}\left(\left|X'(a^\rho\cdot)\right| + \left|X'(b^\rho,\cdot)\right|\right)\left(\frac{1}{(\alpha+s+1)} + B(\alpha, s+1)\right)$$

Proof. Letting $m = 1$, $h_1(t) = t^s, h_2(t) = (1-t)^s$ for all $t \in [0,1]$ and $s \in (0,1]$ in Theorem 3.1 it is had that

$$\mathtt{I}(h_1) = \int_0^1 t^{\rho(\alpha+s+1)-1}dt = \frac{1}{\rho(\alpha+s+1)}, \mathtt{I}(h_2) = \int_0^1 t^{\alpha\rho-1}(1-t^\rho)^s dt = \frac{1}{\rho}B(\alpha, s+1).$$

Replacing these values in (14) it is attained

$$\left|\frac{X(a^\rho,\cdot)+X(b^\rho,\cdot)}{2\alpha\rho} - \frac{\Gamma(\alpha)}{2\rho^{1-\alpha}(b^\rho-a^\rho)^\alpha}\left({}^\rho I^\alpha_{b^\rho-}X(a^\rho,\cdot) + {}^\rho I^\alpha_{a^\rho+}X(b^\rho,\cdot)\right)\right|$$
$$\leq \frac{(b^\rho-a^\rho)}{\rho\alpha}\left(\left|X'(a^\rho\cdot)\right| + \left|X'(b^\rho,\cdot)\right|\right)\left(\frac{1}{(\alpha+s+1)} + B(\alpha, s+1)\right)$$

■

Corollary 4.4. *Let $\alpha > 0$ and $\rho > 0$. Let $X : [a^\rho, b^\rho] \times \Omega \to \mathbb{R}$ be a mean square differentiable stochastic process with $0 \leq a < b$ and $X(t,\cdot) \in X_c^p(a^\rho, b^\rho)$. If $|X'(t,\cdot)|$ is $P-$convex then the following inequality holds almost everywhere*

$$\left|\frac{X(a^\rho,\cdot)+X(b^\rho,\cdot)}{2\alpha\rho} - \frac{\Gamma(\alpha)}{2\rho^{1-\alpha}(b^\rho-a^\rho)^\alpha}\left({}^\rho I^\alpha_{b^\rho-}X(a^\rho,\cdot) + {}^\rho I^\alpha_{a^\rho+}X(b^\rho,\cdot)\right)\right|$$
$$\leq \frac{(b^\rho-a^\rho)}{\alpha^2\rho}\left(\left|X'(a^\rho\cdot)\right| + \left|X'(b^\rho,\cdot)\right|\right)$$

Proof. Using the scheme presented in Corollary 4.2 it is attained the desired result. ■

5. Conclusions

In the present work some inequalities for $(m, h_1, h_2)-$convex stochastic processes have been established and from these results it has been deduced some previous results found in the works of Agahi H. and Babakhani A. [1], D. Kotrys [24] and Hernández Hernández J.E and Gómez J.F [19]. Some others inequalities can be found from the Theorems established using the Lemma 2.11 related with the Riemann Liouville fractional integral, Hadamard fractional integral and the classical Riemann integral. The authors hope that this work serve to stimulate the advance in this research area.

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