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Fractional Ostrowski type inequalities for functions whose first derivatives are MT-preinvex

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1. Introduction

In 1938, A. M. Ostrowski proved an interesting integral inequality, given by the following theorem

Theorem 1.1. *[17] Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping in the interior I° of I, and $a, b \in I^\circ$, with $a < b$. If $|f'| \leq M$ for all $x \in [a, b]$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]. \tag{1}$$

In recent decades, inequality (1) has attracted much interest from many researchers, a considerable papers have been apperaed on the generalizations, variants and extensions of inequality (1). For more details, we advise reader to [4, 5, 6, 8, 9, 10, 12, 13, 14, 18, 21, 22] and references therein.

Recently, lot of efforts have been made by many mathematicians to generalize the classical convexity. Hanson [3], introduced a new class of generalized convex functions, called invex functions. In [1], the authors gave the concept of preinvex functions which is special case of invexity, and many authors have studied their fundamental properties and their role in optimization, variational inequalities and equilibrium problems, we send back to the reader to [15, 16, 20, 25, 26].

In [24] Tunç established the following Ostrowski type inequalities and midpoint type inequalities for functions whose derivatives are MT-convex functions

Theorem 1.2. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L([a,b])$ and $|f'| \leq M$ with $a,b \in I^\circ$ and $a < b$. If $|f'|$ is MT-convex on $[a,b]$, then one has the following inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{\pi\left[(x-a)^2+(b-x)^2\right]}{4(b-a)} M. \tag{2}$$

Corollary 1.3. *In Theorem 1.2 if we choose $x = \frac{a+b}{2}$, we have*

$$\left| f(\tfrac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{1}{8}\pi(b-a)M. \tag{3}$$

Theorem 1.4. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L([a,b])$ and $|f'| \leq M$ with $a,b \in I^\circ$ and $a < b$. If $|f'|^q$ is MT-convex on $[a,b]$ where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, one has the following inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{M}{(1+p)^{\frac{1}{p}}} \left(\frac{\pi}{8}\right)^{\frac{1}{q}} \frac{(x-a)^2+(b-x)^2}{b-a}. \tag{4}$$

Corollary 1.5. *In Theorem 1.4 if we choose $x = \frac{a+b}{2}$, we have*

$$\left| f(\tfrac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{M\pi^{\frac{1}{q}}}{2^{1+\frac{3}{q}}(1+p)^{\frac{1}{p}}}(b-a). \tag{5}$$

Theorem 1.6. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L([a,b])$ and $|f'| \leq M$ with $a,b \in I^\circ$ and $a < b$. If $|f'|^q$ is MT-convex on $[a,b]$ where $q \geq 1$, one has the following inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{M}{2}\left(\frac{\pi}{2}\right)^{\frac{1}{q}} \frac{(x-a)^2+(b-x)^2}{b-a}. \tag{6}$$

Corollary 1.7. *In Theorem 1.6 if we choose $x = \frac{a+b}{2}$, we have*

$$\left| f(\tfrac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{M}{4}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}(b-a). \tag{7}$$

Motivated by the results cited above, in this paper we establish some new Ostrowski's inequalities for functions whose first derivatives in absolute value are MT-preinvex via Reimann-Liouville integral operators.

2. Preliminaries

In this section we recall some concepts of convexity that are well known in the literature. Throughout this section I is an interval of $\mathbb{R}$.

Definition 2.1. *[19] A function $f : I \to \mathbb{R}$ is said to be convex, if*

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and all $t \in [0,1]$.

Definition 2.2. *[23] A nonnegative function $f : I \to \mathbb{R}$ is said to be MT-convex, if the following inequality*

$$f(tx + (1-t)y) \le \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y)$$

holds for all $x, y \in I$ and $t \in (0,1)$.

Let K be a subset in $\mathbb{R}$ and let $f : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}$ be continuous functions.

Definition 2.3. *[25] A set K is said to be invex at x with respect to η, if*

$$x + t\eta(y,x) \in K$$

holds for all $x, y \in K$ and $t \in [0,1]$.

K is said to be an invex set with respect to η if K is invex at each $x \in K$.

Definition 2.4. *[25] A function f on the invex set K is said to be preinvex with respect to η, if*

$$f(x + t\eta(y,x)) \le (1-t)f(x) + tf(y)$$

holds for all $x, y \in K$ and $t \in [0,1]$.

Definition 2.5. *[27] A nonnegative function f on the invex set $K \subseteq [0,\infty)$ is said to be MT-preinvex with respect to η, if*

$$f(x + t\eta(y,x)) \le \frac{\sqrt{1-t}}{2\sqrt{t}} f(x) + \frac{\sqrt{t}}{2\sqrt{1-t}} f(y)$$

holds for all $x, y \in K$ and $t \in (0,1)$.

Definition 2.6. *[7] Let $f \in L[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ of order $\alpha > 0$ with $a \ge 0$ are defined by*

$$\begin{aligned} J^{\alpha}_{a^+}f(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a \\ J^{\alpha}_{b^-}f(x) &= \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x \end{aligned}$$

respectively, where $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$, is the Gamma function and $J^0_{a^+}f(x) = J^0_{b^-}f(x) = f(x)$.

Definition 2.7. *[2] The incomplete beta function is defined as follows:*

$$B_x(\alpha,\beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt,$$

where $x \in [0,1]$ *and* $\alpha,\beta > 0$.

We also give some properties of incomplete beta function

$$B_x(\alpha,\beta) + B_{1-x}(\beta,\alpha) = B(\alpha,\beta), \tag{8}$$

$$B_x(\alpha+1,\beta) + B_x(\alpha,\beta+1) = B_x(\alpha,\beta), \tag{9}$$

$$B_x(\alpha,\beta+1) = \frac{\beta}{\alpha+\beta}B_x(\alpha,\beta) + \frac{x^\alpha(1-x)^\beta}{\alpha+\beta}, \tag{10}$$

and

$$B_x(\alpha+1,\beta) = \frac{\alpha}{\alpha+\beta}B_x(\alpha,\beta) - \frac{x^\alpha(1-x)^\beta}{\alpha+\beta}. \tag{11}$$

Remark 2.8. *From the properties of incomplete beta function and using the fact that* $B\left(\frac{1}{2},\frac{1}{2}\right) = \pi$, *we can easily prove that* $B_{\frac{1}{2}}\left(\frac{3}{2},\frac{3}{2}\right) = \frac{\pi}{8}$, $B_{\frac{1}{2}}(\frac{5}{2},\frac{1}{2}) = \frac{3}{8}\pi - \frac{1}{2}$, $B_{\frac{1}{2}}\left(\frac{1}{2},\frac{3}{2}\right) = \frac{1}{4}\pi + \frac{1}{2}$ *and* $B_{\frac{1}{2}}\left(\frac{3}{2},\frac{1}{2}\right) = \frac{1}{4}\pi - \frac{1}{2}$.

Lemma 2.9. *[11] Let* $f : [a, a+\eta(b,a)] \to \mathbb{R}$ *be a differentiable function with* $a < a+\eta(b,a)$. *If* $f' \in L([a, a+\eta(b,a)])$, *then the following equality for fractional integrals*

$$\begin{aligned} &\left(\left(\frac{x-a}{\eta(b,a)}\right)^\alpha + \left(1-\frac{x-a}{\eta(b,a)}\right)^\alpha\right)f(x) - \frac{\Gamma(\alpha+1)}{(\eta(b,a))^\alpha}\left(J^\alpha_{x^-}f(a) + J^\alpha_{x^+}f(a+\eta(b,a))\right) \\ = \;& \eta(b,a)\left(\int_0^{\frac{x-a}{\eta(b,a)}} t^\alpha f'(a+t\eta(b,a))dt - \int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^\alpha f'(a+t\eta(b,a))dt\right) \end{aligned} \tag{12}$$

holds for all $x \in [a, a+\eta(b,a)]$.

3. Main Results

In what follows $\eta : K \times K \to \mathbb{R}$, and $K \subset \mathbb{R}$ an invex subset with respect to η, and $a, b \in K^\circ$ the interior of K such that $[a, a+\eta(b,a)] \subset K$.

Theorem 3.1. *Let* $f : [a, a+\eta(b,a)] \to \mathbb{R}$ *be a differentiable function such that* $\eta(b,a) > 0$ *and* $f' \in L([a, a+\eta(b,a)])$. *If* $|f'|$ *is MT-preinvex, then the following inequality for fractional integrals*

$$\begin{aligned} &\left|\left(\left(\frac{x-a}{\eta(b,a)}\right)^\alpha + \left(1-\frac{x-a}{\eta(b,a)}\right)^\alpha\right)f(x) - \frac{\Gamma(\alpha+1)}{(\eta(b,a))^\alpha}\left(J^\alpha_{x^-}f(a) + J^\alpha_{x^+}f(a+\eta(b,a))\right)\right| \\ \leq \;& \frac{\eta(b,a)}{2}\left(\left(B_{\frac{x-a}{\eta(b,a)}}\left(\alpha+\frac{1}{2},\frac{3}{2}\right) + B_{1-\frac{x-a}{\eta(b,a)}}\left(\alpha+\frac{3}{2},\frac{1}{2}\right)\right)\left|f'(a)\right|\right. \\ &\left. + \left(B_{\frac{x-a}{\eta(b,a)}}\left(\alpha+\frac{3}{2},\frac{1}{2}\right) + B_{1-\frac{x-a}{\eta(b,a)}}\left(\alpha+\frac{1}{2},\frac{3}{2}\right)\right)\left|f'(b)\right|\right) \end{aligned} \tag{13}$$

holds for all $x \in [a, a+\eta(b,a)]$.

Proof. From Lemma 2.9, properties of modulus, and MT-preinvexity of $|f'|$, we get

$$\begin{aligned}
&\left|\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\alpha}+\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{\alpha}\right)f(x)-\tfrac{\Gamma(\alpha+1)}{(\eta(b,a))^{\alpha}}\left(J_{x^-}^{\alpha}f(a)+J_{x^+}^{\alpha}f(a+\eta(b,a))\right)\right|\\
\leq\ & \eta(b,a)\left(\int_0^{\frac{x-a}{\eta(b,a)}} t^{\alpha}\left|f'(a+t\eta(b,a))\right|dt+\int_{\frac{x-a}{\eta(b,a)}}^{1}(1-t)^{\alpha}\left|f'(a+t\eta(b,a))\right|dt\right)\\
\leq\ & \tfrac{\eta(b,a)}{2}\left(\int_0^{\frac{x-a}{\eta(b,a)}}\left(\left(t^{\alpha-\frac{1}{2}}(1-t)^{\frac{1}{2}}\left|f'(a)\right|+t^{\alpha+\frac{1}{2}}(1-t)^{-\frac{1}{2}}\left|f'(b)\right|\right)\right)dt\right.\\
&\left.+\int_{\frac{x-a}{\eta(b,a)}}^{1}\left((1-t)^{\alpha+\frac{1}{2}}t^{-\frac{1}{2}}\left|f'(a)\right|+(1-t)^{\alpha-\frac{1}{2}}t^{\frac{1}{2}}\left|f'(b)\right|\right)dt\right)\\
=\ & \tfrac{\eta(b,a)}{2}\left(\left(B_{\frac{x-a}{\eta(b,a)}}\left(\alpha+\tfrac{1}{2},\tfrac{3}{2}\right)+B_{1-\frac{x-a}{\eta(b,a)}}\left(\alpha+\tfrac{3}{2},\tfrac{1}{2}\right)\right)\left|f'(a)\right|\right.\\
&\left.+\left(B_{\frac{x-a}{\eta(b,a)}}\left(\alpha+\tfrac{3}{2},\tfrac{1}{2}\right)+B_{1-\frac{x-a}{\eta(b,a)}}\left(\alpha+\tfrac{1}{2},\tfrac{3}{2}\right)\right)\left|f'(b)\right|\right),
\end{aligned}$$

where we have used the facts that

$$\int_0^{\frac{x-a}{\eta(b,a)}} t^{\alpha-\frac{1}{2}}(1-t)^{\frac{1}{2}}\,dt=B_{\frac{x-a}{\eta(b,a)}}\left(\alpha+\tfrac{1}{2},\tfrac{3}{2}\right), \tag{14}$$

$$\int_{\frac{x-a}{\eta(b,a)}}^{1} t^{-\frac{1}{2}}(1-t)^{\alpha+\frac{1}{2}}\,dt=\int_0^{1-\frac{x-a}{\eta(b,a)}} t^{\alpha+\frac{1}{2}}(1-t)^{-\frac{1}{2}}\,dt=B_{1-\frac{x-a}{\eta(b,a)}}\left(\alpha+\tfrac{3}{2},\tfrac{1}{2}\right), \tag{15}$$

$$\int_0^{\frac{x-a}{\eta(b,a)}} t^{\alpha+\frac{1}{2}}(1-t)^{-\frac{1}{2}}\,dt=B_{\frac{x-a}{\eta(b,a)}}\left(\alpha+\tfrac{3}{2},\tfrac{1}{2}\right) \tag{16}$$

and

$$\int_{\frac{x-a}{\eta(b,a)}}^{1} t^{\frac{1}{2}}(1-t)^{\alpha-\frac{1}{2}}\,dt=\int_0^{1-\frac{x-a}{\eta(b,a)}} t^{\alpha-\frac{1}{2}}(1-t)^{\frac{1}{2}}\,dt=B_{1-\frac{x-a}{\eta(b,a)}}\left(\alpha+\tfrac{1}{2},\tfrac{3}{2}\right). \tag{17}$$

The proof is completed. ■

Corollary 3.2. *In Theorem 3.1, if we choose $x=\frac{2a+\eta(b,a)}{2}$, we obtain the following fractional midpoint inequality*

$$\begin{aligned}
&\left|f\left(\tfrac{2a+\eta(b,a)}{2}\right)\right.\\
&\left.-\tfrac{2^{\alpha-1}\Gamma(\alpha+1)}{(\eta(b,a))^{\alpha}}\left(J^{\alpha}_{\left(\frac{2a+\eta(b,a)}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{2a+\eta(b,a)}{2}\right)^+}f(a+\eta(b,a))\right)\right|\\
\leq\ & \tfrac{\eta(b,a)}{2^{2-\alpha}}\left(B_{\frac{1}{2}}\left(\alpha+\tfrac{1}{2},\tfrac{3}{2}\right)+B_{\frac{1}{2}}\left(\alpha+\tfrac{3}{2},\tfrac{1}{2}\right)\right)\left(\left|f'(a)\right|+\left|f'(b)\right|\right).
\end{aligned}$$

Corollary 3.3. *In Theorem 3.1, if we choose $\alpha = 1$, we obtain the following Ostrowski inequality*

$$\begin{aligned} &\left| f(x) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(t)\,dt \right| \\ \leq\ & \frac{\eta(b,a)}{2} \left(\left(B_{\frac{x-a}{\eta(b,a)}}\left(\tfrac{3}{2}, \tfrac{3}{2}\right) + B_{1-\frac{x-a}{\eta(b,a)}}\left(\tfrac{5}{2}, \tfrac{1}{2}\right) \right) \left| f'(a) \right| \right. \\ &\left. + \left(B_{\frac{x-a}{\eta(b,a)}}\left(\tfrac{5}{2}, \tfrac{1}{2}\right) + B_{1-\frac{x-a}{\eta(b,a)}}\left(\tfrac{3}{2}, \tfrac{3}{2}\right) \right) \left| f'(b) \right| \right). \end{aligned}$$

Moreover if we choose $x = \frac{2a+\eta(b,a)}{2}$, we obtain the following midpoint inequality

$$\left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(t)\,dt \right| \leq \frac{\eta(b,a)}{4} (\pi - 1) \left(\left| f'(a) \right| + \left| f'(b) \right| \right).$$

Theorem 3.4. *Let $f : [a, a + \eta(b,a)] \to \mathbb{R}$ be a differentiable function such that $\eta(b,a) > 0$ and $f' \in L([a, a + \eta(b,a)])$ and let $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $|f'|^q$ is MT-preinvex, then the following inequality for fractional integrals*

$$\begin{aligned} &\left| \left(\left(\frac{x-a}{\eta(b,a)} \right)^\alpha + \left(1 - \frac{x-a}{\eta(b,a)} \right)^\alpha \right) f(x) - \frac{\Gamma(\alpha+1)}{(\eta(b,a))^\alpha} \left(J^\alpha_{x^-} f(a) + J^\alpha_{x^+} f(a + \eta(b,a)) \right) \right| \\ \leq\ & \frac{\eta(b,a)}{2^{\frac{1}{q}} (\alpha p+1)^{\frac{1}{p}}} \left(\left(\frac{x-a}{\eta(b,a)} \right)^{\alpha+\frac{1}{p}} \left(B_{\frac{x-a}{\eta(b,a)}}\left(\tfrac{1}{2}, \tfrac{3}{2}\right) \left| f'(a) \right|^q + B_{\frac{x-a}{\eta(b,a)}}\left(\tfrac{3}{2}, \tfrac{1}{2}\right) \left| f'(b) \right|^q \right)^{\frac{1}{q}} \right. \\ &\left. + \left(1 - \frac{x-a}{\eta(b,a)} \right)^{\alpha+\frac{1}{p}} \left(B_{1-\frac{x-a}{\eta(b,a)}}\left(\tfrac{1}{2}, \tfrac{3}{2}\right) \left| f'(a) \right|^q + B_{1-\frac{x-a}{\eta(b,a)}}\left(\tfrac{3}{2}, \tfrac{1}{2}\right) \left| f'(b) \right|^q \right)^{\frac{1}{q}} \right) \end{aligned}$$

holds for all $x \in [a, a + \eta(b,a)]$.

Proof. From Lemma 2.9, properties of modulus, and Hölder's inequality, we have

$$\begin{aligned} &\left| \left(\left(\frac{x-a}{\eta(b,a)} \right)^\alpha + \left(1 - \frac{x-a}{\eta(b,a)} \right)^\alpha \right) f(x) - \frac{\Gamma(\alpha+1)}{(\eta(b,a))^\alpha} \left(J^\alpha_{x^-} f(a) + J^\alpha_{x^+} f(a + \eta(b,a)) \right) \right| \\ \leq\ & \eta(b,a) \left(\left(\int_0^{\frac{x-a}{\eta(b,a)}} t^{\alpha p} dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{x-a}{\eta(b,a)}} \left| f'(a + t\eta(b,a)) \right|^q dt \right)^{\frac{1}{q}} \right. \\ &\left. + \left(\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^{\alpha p}\, dt \right)^{\frac{1}{p}} \left(\int_{\frac{x-a}{\eta(b,a)}}^1 \left| f'(a + t\eta(b,a)) \right|^q dt \right)^{\frac{1}{q}} \right) \\ =\ & \frac{\eta(b,a)}{(\alpha p+1)^{\frac{1}{p}}} \left(\left(\frac{x-a}{\eta(b,a)} \right)^{\alpha+\frac{1}{p}} \left(\int_0^{\frac{x-a}{\eta(b,a)}} \left| f'(a + t\eta(b,a)) \right|^q dt \right)^{\frac{1}{q}} \right. \\ &\left. + \left(1 - \frac{x-a}{\eta(b,a)} \right)^{\alpha+\frac{1}{p}} \left(\int_{\frac{x-a}{\eta(b,a)}}^1 \left| f'(a + t\eta(b,a)) \right|^q dt \right)^{\frac{1}{q}} \right). \end{aligned} \tag{18}$$

Since $|f'|^q$ is MT-preinvex, we deduce

$$\begin{aligned}
&\left|\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\alpha}+\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{\alpha}\right)f(x)-\tfrac{\Gamma(\alpha+1)}{(\eta(b,a))^{\alpha}}\left(J_{x^-}^{\alpha}f(a)+J_{x^+}^{\alpha}f(a+\eta(b,a))\right)\right|\\
\leq\ & \frac{\eta(b,a)}{2^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\alpha+\frac{1}{p}}\right.\\
&\times\left(\left|f'(a)\right|^q\int_0^{\frac{x-a}{\eta(b,a)}}t^{-\frac{1}{2}}(1-t)^{\frac{1}{2}}\,dt+\left|f'(b)\right|^q\int_0^{\frac{x-a}{\eta(b,a)}}t^{\frac{1}{2}}(1-t)^{-\frac{1}{2}}\,dt\right)^{\frac{1}{q}}\\
&+\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{\alpha+\frac{1}{p}}\\
&\left.\times\left(\left|f'(a)\right|^q\int_{\frac{x-a}{\eta(b,a)}}^{1}t^{-\frac{1}{2}}(1-t)^{\frac{1}{2}}\,dt+\left|f'(b)\right|^q\int_{\frac{x-a}{\eta(b,a)}}^{1}t^{\frac{1}{2}}(1-t)^{-\frac{1}{2}}\,dt\right)^{\frac{1}{q}}\right)\\
=\ & \frac{\eta(b,a)}{2^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\alpha+\frac{1}{p}}\left(B_{\frac{x-a}{\eta(b,a)}}\left(\tfrac{1}{2},\tfrac{3}{2}\right)\left|f'(a)\right|^q+B_{\frac{x-a}{\eta(b,a)}}\left(\tfrac{3}{2},\tfrac{1}{2}\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{\alpha+\frac{1}{p}}\left(B_{1-\frac{x-a}{\eta(b,a)}}\left(\tfrac{1}{2},\tfrac{3}{2}\right)\left|f'(a)\right|^q+B_{1-\frac{x-a}{\eta(b,a)}}\left(\tfrac{3}{2},\tfrac{1}{2}\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right),
\end{aligned}$$

which completes the proof. ■

Corollary 3.5. *In Theorem 3.4, if we choose $x=\frac{2a+\eta(b,a)}{2}$, we obtain the following fractional midpoint inequality*

$$\begin{aligned}
&\left|f\left(\tfrac{2a+\eta(b,a)}{2}\right)\right.\\
&\left.-\tfrac{2^{\alpha-1}\Gamma(\alpha+1)}{(\eta(b,a))^{\alpha}}\left(J^{\alpha}_{\left(\frac{2a+\eta(b,a)}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{2a+\eta(b,a)}{2}\right)^+}f(a+\eta(b,a))\right)\right|\\
\leq\ & \frac{\eta(b,a)}{2^{1+\frac{2}{q}}(\alpha p+1)^{\frac{1}{p}}}\left((\pi+2)\left|f'(a)\right|^q+(\pi-2)\left|f'(b)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Corollary 3.6. *In Theorem 3.4, if we choose $\alpha=1$, we obtain the following Ostrowski inequality*

$$\begin{aligned}
&\left|f(x)-\tfrac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(t)\,dt\right|\\
\leq\ & \frac{\eta(b,a)}{2^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{1+\frac{1}{p}}\left(B_{\frac{x-a}{\eta(b,a)}}\left(\tfrac{1}{2},\tfrac{3}{2}\right)\left|f'(a)\right|^q+B_{\frac{x-a}{\eta(b,a)}}\left(\tfrac{3}{2},\tfrac{1}{2}\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{1+\frac{1}{p}}\left(B_{1-\frac{x-a}{\eta(b,a)}}\left(\tfrac{1}{2},\tfrac{3}{2}\right)\left|f'(a)\right|^q+B_{1-\frac{x-a}{\eta(b,a)}}\left(\tfrac{3}{2},\tfrac{1}{2}\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Moreover if we choose $x = \frac{2a+\eta(b,a)}{2}$, *we obtain the following midpoint inequality*

$$\begin{aligned} &\left| f\left(\tfrac{2a+\eta(b,a)}{2}\right) - \tfrac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(t)\,dt \right| \\ \leq\; & \frac{\eta(b,a)}{2^{1+\frac{2}{q}}(p+1)^{\frac{1}{p}}} \left((\pi+2)\left|f'(a)\right|^q + (\pi-2)\left|f'(b)\right|^q \right)^{\frac{1}{q}}. \end{aligned}$$

Theorem 3.7. *Let* $f : [a, a+\eta(b,a)] \to \mathbb{R}$ *be a differentiable function such that* $\eta(b,a) > 0$ *and* $f' \in L([a, a+\eta(b,a)])$ *and let* $q > 1$. *If* $|f'|^q$ *MT-preinvex, then the following inequality for fractional integrals*

$$\begin{aligned} &\left| \left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\alpha} + \left(1 - \tfrac{x-a}{\eta(b,a)}\right)^{\alpha} \right) f(x) - \tfrac{\Gamma(\alpha+1)}{(\eta(b,a))^{\alpha}} \left(J^{\alpha}_{x^-} f(a) + J^{\alpha}_{x^+} f(a+\eta(b,a)) \right) \right| \\ \leq\; & \frac{\eta(b,a)}{2^{\frac{1}{q}}(\alpha+1)^{1-\frac{1}{q}}} \left(\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{(\alpha+1)\left(1-\frac{1}{q}\right)} \right) \right. \\ & \times \left(B_{\frac{x-a}{\eta(b,a)}}\left(\alpha+\tfrac{1}{2}, \tfrac{3}{2}\right) \left|f'(a)\right|^q + B_{\frac{x-a}{\eta(b,a)}}\left(\alpha+\tfrac{3}{2}, \tfrac{1}{2}\right) \left|f'(b)\right|^q \right)^{\frac{1}{q}} \\ & + \left(1 - \tfrac{x-a}{\eta(b,a)}\right)^{(\alpha+1)\left(1-\frac{1}{q}\right)} \\ & \left. \times \left(B_{1-\frac{x-a}{\eta(b,a)}}\left(\alpha+\tfrac{3}{2}, \tfrac{1}{2}\right) \left|f'(a)\right|^q + B_{1-\frac{x-a}{\eta(b,a)}}\left(\alpha+\tfrac{1}{2}, \tfrac{3}{2}\right) \left|f'(b)\right|^q \right)^{\frac{1}{q}} \right) \end{aligned}$$

holds for all $x \in [a, a+\eta(b,a)]$.

Proof. From Lemma 2.9, properties of modulus, and power mean inequality, we have

$$\begin{aligned} &\left| \left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\alpha} + \left(1 - \tfrac{x-a}{\eta(b,a)}\right)^{\alpha} \right) f(x) - \tfrac{\Gamma(\alpha+1)}{(\eta(b,a))^{\alpha}} \left(J^{\alpha}_{x^-} f(a) + J^{\alpha}_{x^+} f(a+\eta(b,a)) \right) \right| \\ \leq\; & \eta(b,a) \left(\left(\int_0^{\frac{x-a}{\eta(b,a)}} t^{\alpha} dt \right)^{1-\frac{1}{q}} \left(\int_0^{\frac{x-a}{\eta(b,a)}} t^{\alpha} \left|f'(a+t\eta(b,a))\right|^q dt \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^{\alpha} dt \right)^{1-\frac{1}{q}} \left(\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^{\alpha} \left|f'(a+t\eta(b,a))\right|^q dt \right)^{\frac{1}{q}} \right) \\ =\; & \frac{\eta(b,a)}{(\alpha+1)^{1-\frac{1}{q}}} \left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{(\alpha+1)\left(1-\frac{1}{q}\right)} \left(\int_0^{\frac{x-a}{\eta(b,a)}} t^{\alpha} \left|f'(a+t\eta(b,a))\right|^q dt \right)^{\frac{1}{q}} \right. \\ & \left. + \left(1 - \tfrac{x-a}{\eta(b,a)}\right)^{(\alpha+1)\left(1-\frac{1}{q}\right)} \left(\int_{\frac{x-a}{\eta(b,a)}}^1 (1-t)^{\alpha} \left|f'(a+t\eta(b,a))\right|^q dt \right)^{\frac{1}{q}} \right). \end{aligned} \tag{19}$$

Since $|f'|^q$ is MT-preinvex and (14)-(17), we deduce

$$\begin{aligned}
&\left|\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{\alpha}+\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{\alpha}\right)f(x)-\tfrac{\Gamma(\alpha+1)}{(\eta(b,a))^{\alpha}}\left(J^{\alpha}_{x^-}f(a)+J^{\alpha}_{x^+}f(a+\eta(b,a))\right)\right| \\
\leq\;& \frac{\eta(b,a)}{2^{\frac{1}{q}}(\alpha+1)^{1-\frac{1}{q}}}\left(\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{(\alpha+1)\left(1-\frac{1}{q}\right)}\right)\right. \\
&\times\left(\left|f'(a)\right|^q\int_0^{\frac{x-a}{\eta(b,a)}}t^{\alpha-\frac{1}{2}}(1-t)^{\frac{1}{2}}\,dt+\left|f'(b)\right|^q\int_0^{\frac{x-a}{\eta(b,a)}}t^{\alpha+\frac{1}{2}}(1-t)^{-\frac{1}{2}}\,dt\right)^{\frac{1}{q}} \\
&+\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{(\alpha+1)\left(1-\frac{1}{q}\right)} \\
&\left.\times\left(\left|f'(a)\right|^q\int_{\frac{x-a}{\eta(b,a)}}^{1}t^{-\frac{1}{2}}(1-t)^{\alpha+\frac{1}{2}}\,dt+\left|f'(b)\right|^q\int_{\frac{x-a}{\eta(b,a)}}^{1}t^{\frac{1}{2}}(1-t)^{\alpha-\frac{1}{2}}\,dt\right)^{\frac{1}{q}}\right) \\
=\;& \frac{\eta(b,a)}{2^{\frac{1}{q}}(\alpha+1)^{1-\frac{1}{q}}}\left(\left(\left(\tfrac{x-a}{\eta(b,a)}\right)^{(\alpha+1)\left(1-\frac{1}{q}\right)}\right)\right. \\
&\times\left(B_{\frac{x-a}{\eta(b,a)}}\left(\alpha+\tfrac{1}{2},\tfrac{3}{2}\right)\left|f'(a)\right|^q+B_{\frac{x-a}{\eta(b,a)}}\left(\alpha+\tfrac{3}{2},\tfrac{1}{2}\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}} \\
&+\left(1-\tfrac{x-a}{\eta(b,a)}\right)^{(\alpha+1)\left(1-\frac{1}{q}\right)} \\
&\left.\times\left(B_{1-\frac{x-a}{\eta(b,a)}}\left(\alpha+\tfrac{3}{2},\tfrac{1}{2}\right)\left|f'(a)\right|^q+B_{1-\frac{x-a}{\eta(b,a)}}\left(\alpha+\tfrac{1}{2},\tfrac{3}{2}\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right)
\end{aligned}$$

which is the desired result. ■

Corollary 3.8. *In Theorem 3.7, if we choose $x=\frac{2a+\eta(b,a)}{2}$, we obtain the following fractional midpoint inequality*

$$\begin{aligned}
&\left|f\left(\tfrac{2a+\eta(b,a)}{2}\right)\right. \\
&\left.-\tfrac{2^{\alpha-1}\Gamma(\alpha+1)}{(\eta(b,a))^{\alpha}}\left(J^{\alpha}_{\left(\frac{2a+\eta(b,a)}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{2a+\eta(b,a)}{2}\right)^+}f(a+\eta(b,a))\right)\right| \\
\leq\;& \frac{2^{\alpha-1}\eta(b,a)}{2^{\alpha+1-\frac{\alpha}{q}}(\alpha+1)^{1-\frac{1}{q}}}\left(\left(B_{\frac{1}{2}}\left(\alpha+\tfrac{1}{2},\tfrac{3}{2}\right)\left|f'(a)\right|^q+B_{\frac{1}{2}}\left(\alpha+\tfrac{3}{2},\tfrac{1}{2}\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(B_{\frac{1}{2}}\left(\alpha+\tfrac{3}{2},\tfrac{1}{2}\right)\left|f'(a)\right|^q+B_{\frac{1}{2}}\left(\alpha+\tfrac{1}{2},\tfrac{3}{2}\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Corollary 3.9. *In Theorem 3.7, if we choose $\alpha = 1$, we obtain the following Ostrowski inequality*

$$\left| f(x) - \frac{1}{\eta(b,a)} \int_{a}^{a+\eta(b,a)} f(t)\,dt \right|$$
$$\leq \frac{\eta(b,a)}{2} \left(\left(\left(\frac{x-a}{\eta(b,a)} \right)^{2-\frac{2}{q}} \right) \left(B_{\frac{x-a}{\eta(b,a)}} \left(\tfrac{3}{2}, \tfrac{3}{2} \right) \left| f'(a) \right|^q + B_{\frac{x-a}{\eta(b,a)}} \left(\tfrac{5}{2}, \tfrac{1}{2} \right) \left| f'(b) \right|^q \right)^{\frac{1}{q}} \right.$$
$$\left. + \left(1 - \frac{x-a}{\eta(b,a)} \right)^{2-\frac{2}{q}} \left(B_{1-\frac{x-a}{\eta(b,a)}} \left(\tfrac{5}{2}, \tfrac{1}{2} \right) \left| f'(a) \right|^q + B_{1-\frac{x-a}{\eta(b,a)}} \left(\tfrac{3}{2}, \tfrac{3}{2} \right) \left| f'(b) \right|^q \right)^{\frac{1}{q}} \right).$$

Moreover if we choose $x = \frac{2a+\eta(b,a)}{2}$, we obtain the following midpoint inequality

$$\left| f\left(\frac{2a+\eta(b,a)}{2} \right) - \frac{1}{\eta(b,a)} \int_{a}^{a+\eta(b,a)} f(t)\,dt \right|$$
$$\leq \frac{\eta(b,a)}{8} \left(\left(\frac{\pi|f'(a)|^q + (3\pi-4)|f'(b)|^q}{2} \right)^{\frac{1}{q}} + \left(\frac{(3\pi-4)|f'(a)|^q + \pi|f'(b)|^q}{2} \right)^{\frac{1}{q}} \right).$$

Referencias

[1] A. Ben-Israel and B. Mond, What is invexity?, J. Austral. Math. Soc., Ser. B, 28(1986), No. 1, 1-9.

[2] A. R. DiDonato, M.P. Jarnagin: The efficient calculation of the incomplete beta-function ratio for half-integer values of the parameters a, b. Math. Comp. 21 (1967), no. 100, 652–662.

[3] M. A. Hanson, On sufficiency of the Kuhn-Tucker conditions, J. Math. Anal. Appl. 80 (1981) 545-550.

[4] Havva Kavurmacı, M. Emin Özdemir and Merve Avcı, New Ostrowski type inequalityes for m-convex functions and applications, Hacettepe Journal ofMathematics and Statistics, Volume 40 (2) (2011), 135–145.

[5] S. Hussain and S. Qaisar, New fractional integral inequalities of type Ostrowski through generalized convex function. J. Appl. Math. Inform. 36 (2018), no. 1-2, 107–114.

[6] İ. İşcan, Ostrowski type inequalities for functions whose derivatives are preinvex. Bulletin of the Iranian Mathematical Society. Vol. 40 (2014), No. 2, pp. 373-386.

[7] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and applications of fractional differential equations. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[8] W. Liu, Ostrowski type fractional integral inequalities for MT-convex functions. Miskolc Math. Notes 16 (2015), no. 1, 249–256.

[9] B. Meftah, Some New Ostrwoski's Inequalities for Functions Whose nth Derivatives are r-Convex. International Journal of Analysis, 2016, 7 pages

[10] B. Meftah, Ostrowski inequalities for functions whose first derivatives are logarithmically preinvex. Chin. J. Math. (N.Y.) 2016, Art. ID 5292603, 10 pp.

[11] B. Meftah, Fractional Ostrowski type inequalities for functions whose first derivatives are s-preinvex in the second sense. International Journal of Analysis and Applications 15 (2017), no. 2, 146–154.

[12] B. Meftah, Some new Ostrowski's inequalities for n-times differentiable mappings which are quasi-convex. Facta Univ. Ser. Math. Inform. 32 (2017), no. 3, 319–327.

[13] B. Meftah, New Ostrowski's inequalities. Rev. Colombiana Mat. 51(2017), no. 1, 57–69.

[14] B. Meftah, Some new Ostrowski's inequalities for functions whose n^{th} derivatives are logarithmically convex. Ann. Math. Sil. 32 (2018), no. 1, 275–284.

[15] M. A. Noor, Variational-like inequalities, Optimization, 30 (1994), 323-330.

[16] M. A. Noor, Invex equilibrium problems, J. Math. Anal. Appl., 302 (2005), 463-475.

[17] A. M. Ostrowski, Über die Absolutabweichung einer differentiierbaren Funktion von ihrem Integralmittelwert. (German) Comment. Math. Helv. 10 (1937), no. 1, 226–227.

[18] M. E. Özdemir, H. Kavurmaci, E. Set, Ostrowski's type inequalities for (α, m)-convex functions, Kyungpook Math. J., 50(2010), 371-378.

[19] J. Pečarić, F. Proschan and Y. L. Tong, Convex functions, partial orderings, and statistical applications. Mathematics in Science and Engineering, 187. Academic Press, Inc., Boston, MA, 1992.

[20] R. Pini, Invexity and generalized Convexity, Optimization 22 (1991) 513-525.

[21] M. Z. Sarikaya, On the Ostrowski type integral inequality. Acta Math. Univ. Comenian. (N.S.) 79 (2010), no. 1, 129–134.

[22] E. Set, M. E. Özdemir and M. Z. Sarıkaya, New inequalities of Ostrowski's type for s-convex functions in the second sense with applications. Facta Univ. Ser. Math. Inform. 27 (2012), no. 1, 67–82.

[23] M. Tunç, H. Yildirim, On MT-convexity, arXiv:1205.5453v1 [math.CA].

[24] M. Tunç, Ostrowski type inequalities for functions whose derivatives are MT-convex. J. Comput. Anal. Appl. 17 (2014), no. 4, 691–696.

[25] T. Weir and B. Mond, (1988). Pre-invex functions in multiple objective optimization, J. Math. Anal.Appl. 136, 29-38.

[26] X. -M. Yang and D. Li, (2001). On properties of preinvex functions, J. Math. Anal. Appl. 256,229-241.

[27] S. Zheng, T. S. Du, S. S. Zhao and L. Z. Chen, New Hermite-Hadamard inequalities for twice differentiable φ-MT-preinvex functions. J. Nonlinear Sci. Appl. 9 (2016), no. 10, 5648–5660.