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# Construction of M wavelet matrices

# Construcción de matrices de ondículas de rango M

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**Abstract**

This article contains a detailed description of the generalization of sequences of orthogonal wavelets of rank 2 made by Daubechies for the case of M wavelet matrices made by Heller, in where we construct several examples that describe in a friendly way the theory developed by Daubechies and Heller.

*Keywords:*
Wavelets, Multiresolution Analysis, Fourier Transform.

**Resumen**

Este artículo contiene una descripción detallada de la generalización de sucesiones de ondículas ortogonales de rango 2 hecha por Daubechies para el caso de matrices de ondículas de rango M hecha por Heller, en donde construimos varios ejemplos que describe de manera amigable la teoría desarrollada por Daubechies y Heller.

*Palabras claves:*
Ondículas, Análisis multirresolución, Transformada de Fourier.

## 1. Introduction

The waves have had a history marked by many independent discoveries and rediscoveries. They have been introduced in 1984 by Morlet and Grossmann where they introduced for the first time the term in the mathematical language. Ives Meyer in 1985, discovered the first soft orthogonal waves and in 1988 Ingrid Daubechies He constructed the first orthogonal waves with compact support, which became a practical tool.

There is an intrinsic relationship between the ideas of the theory of ondiculas and those existing in algorithms to process signals and images; In a general way, the applications of the ondiculas to the systems of communication are increasingly more relevant, this because it is a mathematical instrument that adapts very well to the classical methods of the analysis to process signals. The objective of the analysis of the signal is to extract the desired information and that is within the object of study. For example, Fourier analysis uses an infinite number of sinusoidal and cosinusoidal waves to interpret sounds, images and other applications. However, the analysis according to the wavelets uses a single fundamental element, which is precisely the wavelet chosen according to the convenience of the problem being studied. The expectation of wavelet theory is its optimization in data compression, which is due to its ability to condense in good form the information coming from the signals. For example, the images are decomposed at the level of details; Each part of the image contains information about the other parts.

The algorithms developed by Burt and Adelson in 1983 decompose a signal into its trend and its details using a pair of filters that capture different properties of the signal (see [BA]). In 1987 Y. Meyer and S. Mallat described the algorithms mentioned above in terms of a structure called Multiresolution Analysis where the decomposition into trend and details was manifested in the invariance due to dyadic expansions of the new structure (see [Ma]).

Wavelet theory can be defined as an alternative to classical Fourier theory and aims to construct an orthonormal basis of $L^2(\mathbf{R})$ from a single function by dilatations and translations.

In a first work see [RV] we collected information and developed a generalization for the case of M-wavelets with M>1, from the construction of 2-wavelets from a multiresolution analysis made by Hernández and Weiss in [HW]. In this work we will construct M-wavelet matrices with N vanishing moments that give rise to a scale function that will allow us constructing M-wavelets that have all compact support, where we define the Fourier transform of $f$ as

$$\widehat{f}(\omega) = \int_{\mathbf{R}} f(x)e^{-ix\omega}dx,$$

so we will write the Plancherel theorem

$$\|f\|_2^2 = \frac{1}{2\pi}\|\widehat{f}\|_2.$$

From the computational point of view it is advisable to use filters that are trigonometric polynomials. These give rise to scale functions and M-wavelets of compact support.

In section 2 we will construct M-wavelets with N vanishing moments using the results of [RV], in section 3 we will construct functions of such scale that from them we will develop in section 5 the M-wavelets, with a method developed by P.N. Heller (see [He]), and finally in sections 5 and 6 we will develop several examples of scaling sequence and M-wavelet matrices.

## 2. M-wavelets with N vanishing moments

In [RV] it was shown that to construct M-wavelets $\{\psi^1, \psi^2, ..., \psi^{M-1}\}$ from a M-AMR with scale function $\varphi$ it is enough to find the coefficients $\{a_{0,k} : k \in \mathbf{Z}\}$ of the low-pass filter $m_0$ and the coefficients $\{a_{s,k} : k \in \mathbf{Z}, s = 1, 2, ..., M-1\}$ of high-pass filters $m_1, m_2, ..., m_{M-1}$. These coefficients must satisfy

$$\sum_{k\in\mathbf{Z}} a_{s,k}\overline{a_{s',k+Ml}} = M\delta_{s,s'}\delta_{0,l} \quad l \in \mathbf{Z}, s, s' \in \{0, 1, ..., M-1\} \tag{2.1}$$

Instead of $\widehat{\varphi}(M\omega) = m_0(e^{i\omega})\widehat{\varphi}(\omega)$ it follows that $m_0(e^{i\omega})|_{\omega=0} = 1$ and this implies that $\sum_{k\in\mathbf{Z}} a_{0,k} = M$.

When M-wavelets are construct $\{\psi^1, \psi^2, ..., \psi^{M-1}\}$ continuous and with support compact we should have to $\int_{\mathbf{R}} \psi^{(s)}(x)dx = 0$ , $s = 1, 2, ..., M-1$ (see Theorem 4.1 in [RV]). Therefore of $\widehat{\psi^{(s)}}(0) = 0$ and of $\widehat{\psi^s}(M\omega) = m_s(e^{i\omega})\widehat{\varphi}(\omega)$ it follows that $\sum_{k\in\mathbf{Z}} a_{s,k} = 0$ , $s = 1, 2, ..., M-1$. Therefore, we will assume that

$$\sum_{k\in\mathbf{Z}} a_{s,k} = M\delta_{s,0}, \quad s = 0, 1, 2, ..., M-1 \tag{2.2}$$

**Definition 2.1.** *We will say that the matrix*

$$A = \left(a_{s,k}\right)_{s=0\ k\in\mathbf{Z}}^{M-1}$$

*of order $M \times \infty$ is a* **M-wavelet matrix** *whether satisfy (2.1) and (2.2).*

Our goal is to construct M-wavelet matrix that generate M-wavelets $\{\psi^1, \psi^2, ..., \psi^{M-1}\}$ with a number N of fixed vanishing moments. The condition (2.2) it tells us that each $\psi^s$ has its first vanishing moment. If we want $\int_{\mathbf{R}} x\psi^s(x)dx = 0$ (N=1) for each $s = 1, ..., M-1$ it must have to

$$\frac{d\widehat{\psi^{(s)}}}{d\omega}(0) = (-i)\int_{\mathbf{R}} x\psi^s(x)dx = 0; \tag{2.3}$$

as $\widehat{\psi^{(s)}}(M\omega) = m_s(e^{i\omega})\widehat{\varphi}(\omega)$ taking derivatives and using $m_s(0) = 0$ it is deduced

$$\frac{d\widehat{\psi^{(s)}}}{d\omega}(0) = \frac{dm_s(e^{i\omega})}{d\omega}\Big|_{\omega=0} .\widehat{\varphi}(0) \qquad s = 1, 2, .., M-1. \tag{2.4}$$

Of (2.3) and (2.4) it is deduced (using $\widehat{\varphi}(0) = 1$)

$$0 = \frac{dm_s(e^{i\omega})}{d\omega}\Big|_{\omega=0} = \frac{1}{M}\sum_{k\in\mathbf{Z}}(-ik)a_{s,k} \qquad s = 1, 2, ..., M-1$$

Proceeding by induction, we conclude that the M-wavelets $\{\psi^1, \psi^2, ..., \psi^{M-1}\}$ have the N first vanishing moments when

$$\sum_{k\in\mathbf{Z}} k^n a_{s,k} = 0 \qquad n = 0, 1, ..., N-1 \quad , s = 1, 2, ..., M-1 \tag{2.5}$$

and this condition is equivalent to

$$\frac{d^n m_s(e^{i\omega})}{d\omega^n}\Big|_{\omega=0} = 0 \qquad n = 0, 1, ..., N-1 \quad , s = 1, 2, ..., M-1 \tag{2.6}$$

where $m_s(e^{i\omega}) = \frac{1}{M}\sum_{k\in\mathbf{Z}} a_{s,k}e^{-i\omega k}$.

Therefore, the M-wavelet matrix $A$ has N **vanishing moments** if it is true (2.5) or equivalently (2.6).

**Theorem 2.1.** *Let $A = (a_{s,k})_{s=0\ ,k\in\mathbf{Z}}^{M-1}$ be the M-wavelet matrix. The following conditions are equivalent:*

(i) *The matrix A has N vanishing moments*

(ii) *The low-pass filter $m_0(e^{i\omega})$ has a zero of order N in each of the M th roots of unity $\zeta^m = e^{\frac{2\pi i m}{M}}$ , $m = 1, 2, ..., M-1$.*

**Proof.** (ii)⇒(i) We have to $m_0^{(n)}(\zeta^m) = 0$,for $m = 1, 2, ..., M-1$ and $n = 0, 1, ..., N-1$. We want to try (2.6). By (3.1) we have that

$$\sum_{m=0}^{M-1} m_0(\zeta^m e^{i\omega})\overline{m_s(\zeta^m e^{i\omega})} = 0 \quad , s = 1, 2, ..., M-1. \tag{2.7}$$

Using the hypothesis for $n = 0$ it follows $m_0(e^{i\omega})|_{\omega=0}\overline{m_s(e^{i\omega})}|_{\omega=0} = 0$. Since $m_0(e^{i\omega})|_{\omega=0} = 1$ it follows that

$$m_s(e^{i\omega})|_{\omega=0} = 0 \tag{2.8}$$

For $n = 1$, we derive (2.7) getting

$$\sum_{m=0}^{M-1}\Big[\frac{dm_0(\zeta^m e^{i\omega})}{d\omega}\overline{m_s(\zeta^m e^{i\omega})} + m_0(\zeta^m e^{i\omega})\frac{d\overline{m_s(\zeta^m e^{i\omega})}}{d\omega}\Big] = 0$$

from which we deduce that

$$m_s^{(1)}(e^{i\omega})|_{\omega=0} = 0$$

Suppose that our thesis is fulfilled until the derivative $n-1$. Let us then calculate the nth derivative using the rule of Leibniz that comes given by:

$$(f(x)g(x))^{(n)} = \sum_{k=0}^{n}\binom{n}{k} f^{(n-k)}(x)g^{(k)}(x)$$

So we have that for $s \neq 0$

$$\begin{aligned} 0 = \left(\sum_{m=0}^{M-1} m_0(\zeta^m e^{i\omega})\overline{m_s(\zeta^m e^{i\omega})}\right)^{(n)} &= \sum_{n=0}^{M-1}\left(m_0(\zeta^m e^{i\omega})\overline{m_s(\zeta^m e^{i\omega})}\right)^{(n)} \\ &= \sum_{m=0}^{M-1}\sum_{k=0}^{n}\binom{n}{k} m_0^{(n-k)}(\zeta^m e^{i\omega})\overline{m_s^{(k)}(\zeta^m e^{i\omega})}. \end{aligned}$$

doing $\omega = 0$, using the hypothesis of induction and (ii) we have

$$\begin{aligned} 0 &= \sum_{m=0}^{M-1} m_0(\zeta^m)\overline{m_s^{(n)}(\zeta^m)} \\ &= m_0(1)\overline{m_s^{(n)}(1)}. \end{aligned}$$

Since $m_0(1) = 1$ it follows that $m_s^{(n)}(e^{i\omega})|_{\omega=0} = 0$ with $s = 1, 2, ..., M-1$.

(i)⇒ (ii) We know that

$$\sum_{s=0}^{M-1} |m_s(e^{i\omega})|^2 = 1 \quad \text{and} \quad \sum_{m=0}^{M-1} |m_0(e^{i(\omega+\frac{2\pi m}{M})})|^2 = 1, \tag{2.9}$$

doing $\omega = 0$ we have to

$$\sum_{s=0}^{M-1} |m_s(1)|^2 = 1 \quad \text{and} \quad \sum_{m=0}^{M-1} |m_0(\zeta^m)|^2 = 1. \tag{2.10}$$

From (2.10) we deduce

$$1 = |m_0(1)|^2 + \sum_{m=1}^{M-1} |m_0(\zeta^m)|^2.$$

Therefore

$$m_0(\zeta^m) = 0 \qquad \text{for} \quad m = 1, ..., M-1. \tag{2.11}$$

what proves the case $n = 0$ of our thesis.

From (2.10) we deduce that

$$1 = \sum_{s=0}^{M-1} |m_s(1)|^2 = |m_0(1)| + \sum_{s=1}^{M-1} |m_s(1)|^2,$$

thus

$$\sum_{s=1}^{M-1} |m_s(1)|^2 = 0$$

and therefore

$$m_s(1) = 0 \qquad \text{for} \quad s = 1, ..., M-1. \tag{2.12}$$

From (2.11) and (2.12) we deduce that

$$M(1) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & m_1(\zeta) & \dots & m_1(\zeta^{M-1}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & m_{M-1}(\zeta) & \dots & m_{M-1}(\zeta^{M-1}) \end{bmatrix}.$$

Since $M(1)$ is an orthogonal matrix, it follows that the vectors

$$(m_1(\zeta), ..., m_1(\zeta^{M-1})) \in \mathbf{R^{M-1}}, ..., (m_{M-1}(\zeta), ..., m_{M-1}(\zeta^{M-1})) \in \mathbf{R^{M-1}}$$

are linearly independent and therefore generate $\mathbf{R^{M-1}}$. For (2.11) we have to $m_0(\zeta^m) = 0$ for $m = 1, ..., M-1$; lack to prove that

$$m_0^{(n)}(\zeta^m) = 0 \qquad \text{for} \quad m = 1, ..., M-1, n = 1, ..., N-1.$$

By induction, suppose that $m_0^{(k)}(\zeta^m) = 0, m = 1, ..., M-1$ and $0 \leq k < n$. Then

$$0 = \sum_{m=0}^{M-1} m_0(\zeta^m e^{i\omega})\overline{m_s(\zeta^m e^{i\omega})} \quad \text{when} \quad s = 1, ..., M-1.$$

The $n$-th derivative of this expression and applying the rule of Leibniz gives us:

$$0 = \sum_{m=0}^{M-1} \Big\{ \sum_{j=0}^{n} \binom{n}{j} m_0^{(j)}(\zeta^m e^{i\omega})\overline{m_s^{(n-j)}(\zeta^m e^{i\omega})} \Big\}.$$

Making $\omega = 0$, using the hypothesis of induction and (i) in (2.11) we have to

$$\begin{aligned} 0 &= \sum_{m=0}^{M-1} \Big\{ \sum_{j=0}^{n} \binom{n}{j} m_0^{(j)}(\zeta^m) \overline{m_s^{(n-j)}(\zeta^m)} \Big\} \\ &= \sum_{j=0}^{n} \binom{n}{j} m_0^{(j)}(1) \overline{m_s^{(n-j)}(1)} + \sum_{m=1}^{M-1} m_0^{(n)}(\zeta^m) \overline{m_s(\zeta^m)} \quad \text{(by induction)} \\ &= \sum_{m=1}^{M-1} m_0^{(n)}(\zeta^m) \overline{m_s(\zeta^m)} \quad \text{(by (2.11))}. \end{aligned}$$

If we do $v = (m_0^{(n)}(\zeta), ..., m_0^{(n)}(\zeta^{M-1}))$ we have to

$$< v, v_s >= 0 \quad \forall s = 1, ..., M-1.$$

Consequently $v = 0$ since $\{v_s\}_{s=1}^{M-1}$ is a base for $\mathbf{R^{M-1}}$. So,

$$m_0^{(n)}(\zeta) = 0, ..., m_0^{(n)}(\zeta^{M-1}) = 0$$

■

## 3. Construction of scaling sequence

As we mentioned earlier, our goal is to construct an M-wavelet matrix $A$ with N vanishing moments and that give rise to a function of scale $\varphi$ and to M-wavelets $\{\psi^1, ..., \psi^{M-1}\}$ that have all compact support.

We will start constructing a succession of finite scale $\{a_{0,k} : 0 \le k \le \overline{k}\} \subset \mathbf{R}$ what should satisfy

$$\sum_{k=0}^{\overline{k}} a_{0,k} a_{0,k+Ml} = M\delta_{0,l} \quad l \in \mathbf{Z} \tag{3.1}$$

$$\sum_{k=0}^{\overline{k}} a_{0,k} = M \tag{3.2}$$

and

$$m_0^{(n)}(\zeta^m) = 0 \quad , n = 0, 1, ..., N-1, \; m = 1, 2, ..., M-1 \tag{3.3}$$

where $\zeta^m = e^{i\frac{2\pi m}{M}}$ are the M-th roots of the unit. The condition (3.1) is followed by (2.1), (3.2) is followed by (2.2), and (3.3) is the condition (ii) of Theorem 2.1.

Since the scale succession is finite, $m_0(e^{i\omega})$ is a trigonometric polynomial (of degree $\overline{k}$). $\zeta^m$ is a zero of order N of $m_0(e^{i\omega}), m = 1, 2, ..., M-1$ (see (3.3)), thus we have that

$$m_0(e^{i\omega}) = \Big( \prod_{m=1}^{M-1} \frac{(e^{i\omega} - \zeta^m)}{M} \Big)^N Q(e^{i\omega}), \tag{3.4}$$

where $Q(e^{i\omega})$ is a trigonometric polynomial. Since $\zeta^m$, $m = 1, 2, ..., M$ are the M-th roots of the unit other than 1, we have that

$$\prod_{m=1}^{M-1}(e^{i\omega} - \zeta^m) = \frac{e^{iM\omega} - 1}{e^{i\omega} - 1}. \tag{3.5}$$

We will construct $P(e^{i\omega}) = |m_0(e^{i\omega})|^2$, to later obtain $m_0(e^{i\ omega})$ using Fejer's factorization (Lemma 3.16 of chapter 2 of [HW]). So

$$P(e^{i\omega}) = [H(e^{i\omega})]^N R_N(e^{i\omega})$$

where

$$H(e^{i\omega}) = \Big| \frac{e^{iM\omega} - 1}{M(e^{i\omega} - 1)} \Big|^2 \quad \text{and} \quad R_N(e^{i\omega}) = |Q(e^{i\omega})|^2. \tag{3.6}$$

The orthogonality conditions (3.1), (3.2) and (3.3) are equivalent to

$$P(e^{i\omega}) + P(e^{i(\omega + \frac{2\pi}{M})}) + ... + P(e^{i(\omega + \frac{2\pi(M-1)}{M})}) = 1 \tag{3.7}$$

and $P$ has a zero of order 2N in $\omega$ where $\omega = \frac{2\pi m}{M}$ with $m = 1, 2, ..., M-1$, thus

$$P(e^{i\omega}) = 1 + O(|\omega|^{2N}), \quad \text{in} \quad \omega \approx 0; \tag{3.8}$$

as the coefficients $a_{0,k}$ are real, $P(e^{i\omega})$ and $H(e^{i\omega})$ are pairs. Therefore, $R_N(e^{i\omega})$ is also par. Let's try to find a polynomial of cosines of the shape

$$R_N(x) = \sum_{n=0}^{N-1} \rho_n cosn\omega.$$

Doing $x = cos\omega$ we can write your Taylor expansion around $x = 1$:

$$R_N(x) = \sum_{n=0}^{N-1} r_n(1-x)^n \quad \text{with} \quad r_n = \frac{1}{n!}R_N^{(n)}(x)|_{x=1}.$$

On the other hand,

$$R_N(x) = P(x)[H(x)]^{-N}.$$

Leibniz's rule gives us

$$R_N^{(n)}(x)|_{x=1} = \sum_{k=0}^{n}\binom{n}{k}\Big[(\frac{d}{dx})^k P(x)\Big]_{x=1}\Big[(\frac{d}{dx})^{n-k}[H(x)]^{-N}\Big]_{x=1}. \tag{3.9}$$

Since $P - 1$ has an order zero of 2N in $x = 1$ by (3.8), the expression (3.9) is simplified to:

$$R_N^{(n)}(x)|_{x=1} = \Big[(\frac{d}{dx})^n[H(x)]^{-N}\Big]_{x=1}.$$

Then

$$R_N(x) = \sum_{n=0}^{N-1}\Big[\frac{1}{n!}(\frac{d}{dx})^n[H(x)]^{-N}\Big]_{x=1}(x-1)^n.$$

In other words, $R_N$ matches the first N terms of the Taylor expansion of $H^{-N}$ around $x = 1$, and this we can calculate from the definition of $H$ given in (3.6). We have

$$H(e^{i\omega}) = \frac{1}{M^2}\prod_{m=1}^{M-1}|e^{i\omega} - \zeta^m|^2.$$

If M **is even** we make $M_1 = \frac{M}{2}$. As $\zeta^{M_1} = e^{\frac{2\pi i M_1}{M}} = e^{\pi i} = -1$ and $\zeta^{M_1+1}, ..., \zeta^{2M_1-1}$ are the conjugates of $\zeta^{M_1-1}, ..., \zeta$ respectively, we have that

$$\begin{aligned}
\frac{1}{M^2}\prod_{m=1}^{M-1}|e^{i\omega} - \zeta^m|^2 &= \frac{1}{M^2}|e^{i\omega}+1|^2\prod_{m=1}^{M_1-1}|(e^{i\omega}-\zeta^m)(e^{i\omega}-\bar{\zeta}^m)|^2 \\
&= \frac{1}{M^2}|e^{i\omega}+1|^2\prod_{m=1}^{M_1-1}|(1-e^{-i\omega}e^{\frac{2\pi i m}{M}})(1-e^{-i\omega}e^{\frac{-2\pi i m}{M}})|^2 \\
&= \frac{1}{M^2}|e^{\frac{i\omega}{2}}(e^{\frac{i\omega}{2}}+e^{-\frac{i\omega}{2}})|^2\prod_{m=1}^{M_1-1}|(1-e^{i(\omega+\frac{2\pi m}{M})})(1-e^{i(\omega-\frac{2\pi m}{M})})|^2 \\
&= \frac{1}{M^2}4cos^2\frac{\omega}{2}\prod_{m=1}^{M_1-1}|e^{i\omega}(e^{-i\omega}-e^{i\frac{2\pi m}{M}}-e^{-i\frac{2\pi m}{M}}+e^{i\omega})|^2 \\
&= \frac{2}{M^2}(1+cos\omega)\prod_{m=1}^{M_1-1}\Big|2cos\omega-2cos\frac{2\pi m}{M}\Big|^2. \qquad (3.10)
\end{aligned}$$

Therefore

$$\begin{aligned}
H(cos\omega) &= \frac{2}{M^2}(1+cos\omega)\prod_{m=1}^{M_1-1}4\Big(cos\omega-cos\frac{2\pi m}{M}\Big)^2 \\
&= \frac{2^M 2^{-1}}{M^2}(1+cos\omega)\prod_{m=1}^{M_1-1}(cos\omega-cos\frac{2\pi m}{M})^2.
\end{aligned}$$

Doing $x = cos\omega$, we have that

$$H(x) = \frac{2^{M-1}}{M^2}(x+1)\prod_{m=1}^{M_1-1}(x-cos\frac{2\pi m}{M})^2 \quad \text{with M } \textbf{even}. \qquad (3.11)$$

On the other hand, for M **odd** we make $M_1 = \frac{M-1}{2}$ and we have

$$\begin{aligned}
\frac{1}{M^2}\prod_{m=1}^{M-1}|e^{i\omega}-\zeta^m|^2 &= \frac{1}{M^2}\prod_{m=1}^{2M_1}|e^{i\omega}-\zeta^m|^2 \\
&= \frac{1}{M^2}\prod_{m=1}^{M_1}|(e^{i\omega}-\zeta^m)(e^{i\omega}-\bar{\zeta}^m)|^2
\end{aligned}$$

By an analogous procedure to the previous one, it is deduced

$$\begin{aligned}\frac{1}{M^2}\prod_{m=1}^{M-1}|e^{i\omega}-\zeta^m|^2 &= \frac{1}{M^2}\prod_{m=1}^{M_1}\Big|(1-e^{i(\omega+\frac{2\pi m}{M})})(1-e^{i(\omega-\frac{2\pi m}{M})})\Big|^2\\ &= \frac{1}{M^2}\prod_{m=1}^{M_1}4\Big(cos\omega-cos\frac{2\pi m}{M}\Big)^2.\end{aligned}$$

And doing $x = cos\omega$,

$$H(x)=\frac{2^{M-1}}{M^2}\prod_{m=1}^{M_1}\Big(x-cos\frac{2\pi m}{M}\Big)^2 \qquad \text{with M } \textbf{odd}. \tag{3.12}$$

Consider the expansion in series of powers of $[H(x)]^{-N}$.

For

$$f(x)=\Big(x-cos\frac{2\pi m}{M}\Big)^{-2N},$$

his Taylor series around $x = 1$ is

$$\Big(x-cos\frac{2\pi m}{M}\Big)^{-2N}=\sum_{n=0}^{\infty}\binom{2N+n-1}{2N-1}(-1)^n\Big(1-cos\frac{2\pi m}{M}\Big)^{-2N-n}(x-1)^n. \tag{3.13}$$

The Taylor series of $(x+1)^{-N}$ around $x = 1$ is

$$(x+1)^{-N}=\sum_{n=0}^{\infty}\binom{N+n-1}{N-1}(-1)^n2^{-N-n}(x-1)^n. \tag{3.14}$$

We use (3.11) together with (3.13) and (3.14) when M **is even** and we obtain that

$$\begin{aligned}[H(x)]^{-N} &= \Big[\frac{M^2}{2^{M-1}}\Big]^N\sum_{n=0}^{\infty}\binom{N+n-1}{N-1}(-1)^n2^{-N-n}(x-1)^n\times\\ &\times\prod_{m=1}^{M_1-1}\sum_{n=0}^{\infty}\binom{2N+n-1}{2N-1}(-1)^n\Big(1-cos\frac{2\pi m}{M}\Big)^{-2N-n}(x-1)^n.\end{aligned} \tag{3.15}$$

Then

$$\begin{aligned}[H(x)]^{-N} &= \Big[\frac{M^2}{2^{M-1}}\Big]^N\sum_{k_{M_1}=0}^{\infty}\binom{N+k_{M_1}-1}{N-1}(1-cos\pi)^{-N-k_{M_1}}(1-x)^{k_{M_1}}\times\\ &\times\sum_{n=0}^{\infty}\sum_{k_1+k_2+\ldots+k_{M_1-1}=n}\prod_{m=1}^{M_1-1}\binom{2N+k_m-1}{2N-1}\Big(1-cos\frac{2\pi m}{M}\Big)^{-2N-k_m}(1-x)^n\\ &= \Big[\frac{M^2}{2^{M-1}}\Big]^N\sum_{n=0}^{\infty}\sum_{k_1+k_2+\ldots+k_{M_1}=n}\Big\{\prod_{m=1}^{M_1-1}\binom{2N+k_m-1}{2N-1}\Big(1-cos\frac{2\pi m}{M}\Big)^{-2N-k_m}\Big\}\times\\ &\times\binom{N+k_{M_1}-1}{N-1}2^{-N-k_{M_1}}(1-x)^n.\end{aligned}$$

Therefore for M **even** we have

$$
\begin{aligned}
r_n &= \left[\frac{M^2}{2^{M-1}}\right]^N \sum_{k_1+k_2+\ldots+k_{M_1}=n} \left\{ \prod_{m=1}^{M_1-1} \binom{2N+k_m-1}{2N-1} \Big(1-cos\frac{2\pi m}{M}\Big)^{-2N-k_m} \right\} \times \\
&\quad \times \binom{N+k_{M_1}-1}{N-1} 2^{-N-k_{M_1}}
\end{aligned}
$$

or

$$
\begin{aligned}
r_n &= \left[\frac{M^2}{2^{M-1}}\right]^N 2^{-N} \Big[ \prod_{m=1}^{M_1-1} \Big(1-cos\frac{2\pi m}{M}\Big)^{-2N} \Big] \\
&\quad \sum_{k_1+k_2+\ldots+k_{M_1}=n} \left\{ \prod_{m=1}^{M_1-1} \binom{2N+k_m-1}{2N-1} \Big(1-cos\frac{2\pi m}{M}\Big)^{-k_m} \right\} \times \\
&\quad \times \binom{N+k_{M_1}-1}{N-1} 2^{-k_{M_1}}.
\end{aligned}
$$

In (3.10) doing $\omega = 0$ we have that

$$
\frac{1}{M^2} \prod_{m=1}^{M-1} |1-\zeta^m|^2 = \frac{2^{M-1}}{M^2} 2 \prod_{m=1}^{M_1-1} (1-cos\frac{2\pi m}{M})^2.
$$

Thus

$$
\prod_{m=1}^{M_1-1} \Big(1-cos\frac{2\pi m}{M}\Big)^{-2N} = \left[\frac{1}{2^{M-1}2}\right]^{-N} \Big[ \prod_{m=1}^{M-1} |1-\zeta^m|^2 \Big]^{-N}.
$$

As

$$
\prod_{m=1}^{M-1} (e^{i\omega}-\zeta^m) = 1 + e^{i\omega} + \ldots + e^{i(M-1)\omega}
$$

for $\omega = 0$ you have to

$$
\prod_{m=1}^{M-1} (1-\zeta^m) = 1 + 1 + \ldots + 1 = M.
$$

Therefore;

$$
\prod_{m=1}^{M_1-1} \Big(1-cos\frac{2\pi m}{M}\Big)^{-2N} = \frac{1}{2^{-MN}} M^{-2N}.
$$

Substituting in $r_n$ we have that

$$\begin{aligned} r_n &= \left[\frac{M^2}{2^{M-1}}\right]^N 2^{-N}\frac{M^{-2N}}{2^{-MN}} \sum_{k_1+k_2+...+k_{M_1}=n} \left\{ \prod_{m=1}^{M_1-1} \binom{2N+k_m-1}{2N-1}\Big(1-cos\frac{2\pi m}{M}\Big)^{-k_m}\right\} \\ &\quad \times \binom{N+k_{M_1}-1}{N-1} 2^{-k_{M_1}} \\ &= \frac{M^{2N}}{2^{MN}2^{-N}} 2^{-N}\frac{M^{-2N}}{2^{-MN}} \sum_{k_1+k_2+...+k_{M_1}=n} \left\{ \prod_{m=1}^{M_1-1} \binom{2N+k_m-1}{2N-1}\Big(1-cos\frac{2\pi m}{M}\Big)^{-k_m}\right\} \\ &\quad \times \binom{N+k_{M_1}-1}{N-1} 2^{-k_{M_1}}. \end{aligned}$$

Consequently,

$$\begin{aligned} r_n &= \sum_{k_1+k_2+...+k_{M_1}=n} \left\{ \prod_{m=1}^{M_1-1} \binom{2N+k_m-1}{2N-1}\Big(1-cos\frac{2\pi m}{M}\Big)^{-k_m}\right\} \\ &\quad \times \binom{N+k_{M_1}-1}{N-1} 2^{-k_{M_1}} \quad \text{with M } \textbf{even}. \end{aligned} \tag{3.16}$$

For M **odd** by (3.12) we have

$$H(x) = \left(\frac{2^{M-1}}{M^2}\right) \prod_{m=1}^{M_1} \Big(x - cos\frac{2\pi m}{M}\Big)^2.$$

Thus

$$[H(x)]^{-N} = \left[\frac{M^2}{2^{M-1}}\right]^N \prod_{m=1}^{M_1} \Big(x - cos\frac{2\pi m}{M}\Big)^{-2N} \qquad \text{with M } \textbf{odd}. \tag{3.17}$$

Then for (3.13)

$$[H(x)]^{-N} = \left[\frac{M^2}{2^{M-1}}\right]^N \prod_{m=1}^{M_1} \sum_{n=0}^{\infty} \binom{2N+n-1}{2N-1}(-1)^n(1-cos\frac{2\pi m}{M})^{-2N-n}(x-1)^n.$$

Then analogously,

$$\begin{aligned} [H(x)]^{-N} &= \left[\frac{M^2}{2^{M-1}}\right]^N \times \\ &\quad \times \sum_{n=0} \sum_{k_1+k_2+...+k_{M_1}=n} \left\{ \prod_{m=1}^{M_1} \binom{2N+k_m-1}{2N-1}\Big(1-cos\frac{2\pi m}{M}\Big)^{-2N-k_m}\right\}(1-x)^n. \end{aligned}$$

Therefore for M **odd** we find

$$\begin{aligned} r_n &= \left[\frac{M^2}{2^{M-1}}\right]^N \sum_{k_1+k_2+...+k_{M_1}=n} \left\{ \prod_{m=1}^{M_1} \binom{2N+k_m-1}{2N-1} \Big(1-cos\frac{2\pi m}{M}\Big)^{-2N-k_m} \right\} \\ &= \left[\frac{M^2}{2^{M-1}}\right]^N \prod_{m=1}^{M_1} \Big(1-cos\frac{2\pi m}{M}\Big)^{-2N} \times \\ \times & \sum_{k_1+k_2+...+k_{M_1}=n} \left\{ \prod_{m=1}^{M_1} \binom{2N+k_m-1}{2N-1} \Big(1-cos\frac{2\pi m}{M}\Big)^{-k_m} \right\}. \end{aligned}$$

In (3.12) doing $\omega = 0$ we have that

$$\begin{aligned} \prod_{m=1}^{M_1} \Big(1-cos\frac{2\pi m}{M}\Big)^{-2N} &= \left[\frac{1}{2^{M-1}}\right]^{-N} \left[\prod_{m=1}^{M-1} |1-\zeta^m|^2\right]^{-N} \\ &= \frac{1}{2^{-MN}2^N} M^{-2N} \\ &= \left[\frac{M^2}{2^{M-1}}\right]^{-N}. \end{aligned}$$

In consecuense

$$r_n = \sum_{k_1+k_2+...+k_{M_1}=n} \left\{ \prod_{m=1}^{M_1} \binom{2N+k_m-1}{2N-1} \Big(1-cos\frac{2\pi m}{M}\Big)^{-k_m} \right\} \qquad \text{with M } \textbf{odd}. \tag{3.18}$$

So we have that $R_N$ is the finite trigonometic polynomial

$$R_N(e^{i\omega}) = \sum_{n=0}^{N-1} r_n(1-cos\omega)^n \tag{3.19}$$

With $r_n$ given by (3.16) for M **even** and (3.18) for M **odd**.

Note that $r_n$ is positive and $cos\omega < 1$ for all $\omega \neq 0$, therefore $R_N$ is a trigonometric polynomial positive.

**Lemma 3.1.** *The solution $P = H^N R_N$, with $R_N$ given by (3.19) and (3.16) if M* **is even** *or (3.18) if M* **is odd** *which satisfies (3.3), also satisfies the orthogonality condition (3.7).*

**Proof.** Define

$$\phi(e^{i\omega}) = P(e^{i\omega}) + P(e^{i(\omega+\frac{2\pi}{M})}) + ... + P(e^{i(\omega+\frac{2\pi(M-1)}{M})}) - 1,$$

then $\phi + 1$ is the periodization of $P$ to the interval $[0, \frac{2\pi}{M}]$.

Since $\phi$ is real, even, and periodic with period $\frac{2\pi}{M}$, it must have the trigonometric polynomial expansion

$$\phi(e^{i\omega}) = \sum_{k=0}^{N-1} c_k(e^{iMk\omega} + e^{-iMk\omega}) \tag{3.20}$$

By construction $\phi$ is a flat of order N in $x = cos\omega$ at $x = 1$; this means that $\phi(x)$ can approximate in a environment of 1 by a polynomial in $(x-1)$ of degree N, and the error of this approximation is of order higher than $(x-1)^N$ when $x \to 1$; in other words

$$\phi(x) \quad = \quad (x-1)^N \quad \text{for} \quad x \approx 1.$$

As $P(e^{i\omega}) = 1 + O(|\omega|^{2N})$ in $\omega = 0$, then $\phi(e^{i\omega})$ is of order $|\omega|^{2N}$ when $\omega \to 0$, thus

$$\phi(e^{i\omega}) \quad = \quad \omega^{2N} \quad \text{for} \quad \omega \approx 0,$$

and

$$\left[\left(\frac{d}{d\omega}\right)^n \phi(e^{i\omega})\right]_{\omega=0} = 0, \quad \text{for} \quad n = 0, 1, ..., 2N-1\ . \tag{3.21}$$

On the other hand, of (3.20) we have that

$$\left(\frac{d}{d\omega}\right)^n \phi(e^{i\omega}) \quad = \quad \sum_{k=0}^{N-1} c_k[(iMk)^n e^{iMk\omega} + (-iMk)^n e^{-iMk\omega}].$$

However,

$$\left[\left(\frac{d}{d\omega}\right)^n \phi(e^{i\omega})\right]_{\omega=0} \quad = \quad \sum_{k=0}^{N-1} c_k[(iMk)^n + (-iMk)^n]$$

$$= \quad \begin{cases} 0 & \text{if n is odd} \\ 2(-M^2)^{\frac{n}{2}} \sum_{k=0}^{N-1} c_k k^n & \text{if n is even,} \quad n \leq 2N-2\ . \end{cases} \tag{3.22}$$

By matching (3.21) and (3.22) you get a system of Vandermonde of order $N \times N$ for the $c_k$, with $k = 0, 1, ..., N-1$:

$$\begin{bmatrix} 1 & 1 & 1 & ... & 1 & 1 & 1 \\ 0 & 1 & 4 & ... & k^2 & ... & (N-1)^2 \\ 0 & 1 & 16 & ... & k^4 & ... & (N-1)^4 \\ 0 & 1 & & & ... & & \\ 0 & 1 & & & ... & & \\ 0 & 1 & & & ... & & \\ 0 & 1 & 2^{2N-2} & ... & k^{2N-2} & ... & (N-1)^{2N-2} \end{bmatrix} \begin{bmatrix} c_o \\ c_1 \\ c_2 \\ . \\ . \\ . \\ c_{N-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}$$

Because it is a Vandermonde system, its associated matrix has a non-zero determinant, which implies that is invertible and therefore this system has a solution only the trivial, that is, $c_0 = c_1 = c_2 = ... = c_{N-1} = 0$. In consequence $\phi(e^{i\omega}) = 0$ what it means

$$P(e^{i(\omega)}) + P(e^{i(\omega + \frac{2\pi}{M})}) + ... + P(e^{i(\omega + \frac{2\pi(M-1)}{M})}) = 1.$$

■

With the Lemma 3.1, we have completed the proof of following theorem

**Theorem 3.1.** *A P solution, which is the module squared of the low-pass filter corresponding to an M-wavelet with N vanishing moments, is given by*

$$P(e^{i\omega}) \quad = \quad \left| \frac{1 + e^{-i\omega} + e^{-i2\omega} + ... + e^{-i(M-1)\omega}}{M} \right|^{2N} R_N(e^{i\omega})$$

*with $R_N$ given by (3.19) and (3.16) if $M$* **is even** *or (3.18) if $M$* **is odd**.

The low-pass filter $m_0$ will be a spectral factor of $P$ of the form

$$m_0(e^{ik\omega}) = \left( \frac{1 + e^{-i\omega} + e^{-i2\omega} + ... + e^{-i(M-1)\omega}}{M} \right)^N Q_N(e^{i\omega}),$$

where the trigonometric polynomial $Q_N$ is a spectral factor of $R_N$. This is calculated using the Fejer-Riesz method ([HW], page 99, section 2.5), finding

$$Q_N(e^{i\omega}) \quad = \quad \sum_{n=0}^{N-1} c_n e^{-in\omega} \quad \text{such that}$$

$$\begin{aligned} R_N(e^{i\omega}) \quad &= \quad \sum_{n=0}^{N-1} b_n cosn\omega \\ &= \quad Q_N(e^{i\omega})\overline{Q_N(e^{i\omega})} \end{aligned}$$

This factorization depends on the fact that $R_N(e^{i\omega}) \geq 0$ for $\omega \in [0,2\pi]$, what that we have observed previously.

## 4. Example of scaling sequence

Now we will use the methods developed previously for to construct scaling sequence of M-AMR that produce M-wavelets with N vanishing moments.

**Example 4.1:** M=3 and N=2 (3-wavelets with 2 vanishing moments). In this case

$$\begin{aligned} P(e^{i\omega}) \quad &= \quad \left| \frac{1 + e^{-i\omega} + e^{-i2\omega}}{3} \right|^{2N} R_N(e^{i\omega}) \\ &= \quad |e^{-i\omega}|^{2N} \left| \frac{e^{i\omega} + e^{-i\omega} + 1}{3} \right|^{2N} R_N(e^{i\omega}) \\ &= \quad \left( \frac{1 + 2cos\omega}{3} \right)^{2N} R_N(e^{i\omega}). \end{aligned}$$

To calculate $R_N$ we use

$$R_N(e^{i\omega}) \quad = \quad \sum_{n=0}^{N-1} r_n (1 - cos\omega)^n$$

with $r_n$ given by the formula (3.18) with $M_1 = 1$ (since $M = 3$):

$$\begin{aligned} r_n &= \binom{2N+n-1}{2N-1}\Big(1 - cos\frac{2\pi}{3}\Big)^{-n} \\ &= \binom{2N+n-1}{2N-1}\Big(\frac{3}{2}\Big)^{-n} . \end{aligned}$$

Therefore we have that

$$R_N(e^{i\omega}) = \sum_{n=0}^{N-1}\binom{2N+n-1}{2N-1}\Big(\frac{2}{3}\Big)^n(1 - cos\omega)^n$$

and

$$P(e^{i\omega}) = \left(\frac{1+2cos\omega}{3}\right)^{2N}\sum_{n=0}^{N-1}\binom{2N+n-1}{2N-1}\Big(\frac{2}{3}\Big)^n(1 - cos\omega)^n \quad \text{for} \quad M = 3 .$$

Making $N$=2,

$$\begin{aligned} R_2(e^{i\omega}) &= \sum_{n=0}^{1}\binom{4+n-1}{4-1}\Big(\frac{2}{3}\Big)^n(1 - cos\omega)^n \\ &= \frac{11}{3} - \frac{8}{3}cos\omega . \end{aligned}$$

We want to find $Q_2(e^i\omega) = a + b^{-i\omega}$ such that

$$|Q_2(e^{i\omega})|^2 = R_2(e^{i\omega});$$

then $\quad (a + b^{i\omega})(a + b^{-i\omega}) = \frac{11}{3} - \frac{8}{3}cos\omega \quad$ if and only if

$$\begin{aligned} \frac{11}{3} - \frac{8}{3}cos\omega &= a^2 + abe^{-i\omega} + abe^{i\omega} + b^2 \\ &= a^2 + b^2 + 2abcos\omega . \end{aligned}$$

Therefore we have the following system:

$$\begin{cases} a^2 + b^2 = \frac{11}{3} \\ 2ab = \frac{-8}{3} . \end{cases}$$

Thus

$$a = \frac{-4}{3b} \qquad and \qquad 9b^4 - 33b^2 + 16 = 0.$$

Resolving this equation of the second degree we have:

$$b = \pm\sqrt{\frac{11 \pm \sqrt{57}}{6}}$$

Therefore,

$$\text{when} \qquad b = \frac{1}{2} + \frac{\sqrt{57}}{6} \quad , \quad a = \frac{1}{2} - \frac{\sqrt{57}}{6}$$

in the same way,

$$\text{when} \qquad b = \frac{1}{2} - \frac{\sqrt{57}}{6} \quad , \quad a = \frac{1}{2} + \frac{\sqrt{57}}{6}$$

Thus

$$Q_2(e^{i\omega}) = \frac{1}{2}\left\{1 \pm \frac{\sqrt{57}}{3} + \left(1 \mp \frac{\sqrt{57}}{3}\right)e^{-i\omega}\right\}$$

Now we can find the succession of scale. We know that $P = HNR_N$, which with $N = 2$ produces

$$P = H^2 R_2 = H^2|Q_2|^2.$$

Therefore, for $M = 3$ and $N = 2$,

$$\begin{aligned}
P(e^{i\omega}) &= \left|\frac{1 + e^{-i\omega} + e^{-i2\omega}}{3}\right|^4 R_2(e^{i\omega}) \\
&= \left|\left(\frac{1 + e^{-i\omega} + e^{-i2\omega}}{3}\right)^2\right|^2 |Q_2(e^{i\omega})|^2 \\
&= \left|\left(\frac{1 + e^{-i\omega} + e^{-i2\omega}}{3}\right)^2\right|^2 |a + be^{-i\omega}|^2 \\
&= \left(\frac{1 + e^{-i\omega} + e^{-i2\omega}}{3}\right)^2 \overline{\left(\frac{1 + e^{-i\omega} + e^{-i2\omega}}{3}\right)^2} (a + be^{-i\omega})\overline{(a + be^{-i\omega})}
\end{aligned}$$

As $P(e^{i\omega}) = m_0(e^{i\omega})\overline{m_0(e^{i\omega})}$ we have that

$$\begin{aligned}
m_0(e^{i\omega}) &= \left(\frac{1 + e^{-i\omega} + e^{-i2\omega}}{3}\right)^2 (a + be^{-i\omega}) \\
&= \frac{1}{3}\left[\frac{a}{3} + \frac{(2a + b)}{3}e^{-i\omega} + \frac{(3a + 2b)}{3}e^{-i2\omega} + \frac{(2a + 3b)}{3}e^{-i3\omega} + \frac{(a + 2b)}{3}e^{-i4\omega} + \frac{b}{3}e^{-i5\omega}\right].
\end{aligned}$$

As $m_0(e^{i\omega}) = \frac{1}{3}\sum_{k=0}^{5} a_{0,k}e^{-ik\omega}$ the coefficients $a_{0,k}$ are given by

$$a_{0,0} = \frac{a}{3} = \frac{1}{3}\left(\frac{1}{2} \pm \frac{\sqrt{57}}{6}\right) = \frac{1}{6}\left[\frac{3 \pm \sqrt{57}}{3}\right] = \left[\frac{3 \pm \sqrt{57}}{18}\right]$$

$$a_{0,1} = \frac{(2a + b)}{3} = \frac{1}{3}\left[2\left[\frac{3 \pm \sqrt{57}}{6}\right] + \left[\frac{3 \mp \sqrt{57}}{6}\right]\right] = \frac{9 \pm \sqrt{57}}{18}$$

Proceeding in this way, we obtain that the scaling sequence for $M = 3$ and $N = 2$ is given by:

$$\left\{a_{0,k}\right\} = \left\{\frac{3 \pm \sqrt{57}}{18}, \frac{9 \pm \sqrt{57}}{18}, \frac{15 \pm \sqrt{57}}{18}, \frac{15 \mp \sqrt{57}}{18}, \frac{9 \mp \sqrt{57}}{18}, \frac{3 \mp \sqrt{57}}{18}\right\}$$

**Example 4.2:** M=4 and N=2 (4-wavelets with 2 vanishing moments). In this case

$$\begin{aligned}
P(e^{i\omega}) &= \left|\frac{1 + e^{-i\omega} + e^{-i2\omega} + e^{-3i\omega}}{4}\right|^{2N} R_N(e^{i\omega}) \\
&= \frac{1}{16^N}\left[(1 + e^{-i\omega} + e^{-i2\omega} + e^{-i3\omega})(1 + e^{i\omega} + e^{i2\omega} + e^{i3\omega})\right]^N R_N(e^{i\omega}) \\
&= \frac{1}{16^N}\left[4 + 6cos\omega + 4cos2\omega + 2cos3\omega\right]^N R_N(e^{i\omega}) \\
&= \left[\frac{1}{2}\left(\frac{1}{2} + \frac{3}{4}cos\omega + \frac{1}{2}cos2\omega + \frac{1}{4}cos3\omega\right)\right]^N R_N(e^{i\omega}).
\end{aligned}$$

Since

$$cos^2\omega = \frac{1}{2} + \frac{cos2\omega}{2} \qquad \text{and} \qquad cos^3\omega = \frac{3}{4}cos\omega + \frac{1}{4}cos3\omega$$

we have that

$$P(e^{i\omega}) = \left[\frac{cos^2\omega + cos^3\omega}{2}\right]^N R_N(e^{i\omega}).$$

Now let's find an explicit form of $R_N(e^{i\omega})$. Doing $M = 4$ ($M_1 = 2$) in (3.16)

$$\begin{aligned}
r_n &= \sum_{k_1+k_2=n} \binom{2N + k_1 - 1}{2N - 1}\left(1 - cos\frac{2\pi}{4}\right)^{-k_1} \times \binom{N + k_2 - 1}{N - 1} 2^{-k_2} \\
&= \sum_{k_2=n-k_1} \binom{2N + k_1 - 1}{2N - 1}\binom{N + n - k_1 - 1}{N - 1} 2^{-n+k_1} \\
&= \sum_{k=0}^{n} \binom{2N + k - 1}{2N - 1}\binom{N + n - k - 1}{N - 1} 2^{k-n}.
\end{aligned}$$

Then by (3.19)

$$R_N(e^{i\omega}) = \sum_{n=0}^{N-1}\sum_{k=0}^{n} \binom{2N + k - 1}{2N - 1}\binom{N + n - k - 1}{N - 1} 2^{k-n}(1 - cos\omega)^n,$$

thus

$$P(e^{i\omega}) = \left[\frac{cos^2\omega + cos^3\omega}{2}\right]^N \times \sum_{n=0}^{N-1}\sum_{k=0}^{n} \binom{2N + k - 1}{2N - 1}\binom{N + n - k - 1}{N - 1} 2^{k-n}(1 - cos\omega)^n.$$

Doing $N$=2 we have that,

$$\begin{aligned}
R_2(e^{i\omega}) &= \sum_{n=0}^{N-1}\sum_{k=0}^{n} \binom{4 + k - 1}{4 - 1}\binom{2 + n - k - 1}{2 - 1} 2^{k-n}(1 - cos\omega)^n \\
&= 1 + (1 - cos\omega) + 4(1 - cos\omega) \\
&= 6 - 5cos\omega.
\end{aligned}$$

We want to find $Q_2(e^i\omega) = a + b^{-i\omega}$ such that

$$|Q_2(e^{i\omega})|^2 = R_2(e^{i\omega}).$$

As $(a + b^{i\omega})(a + b^{-i\omega}) = 6 - 5cos\omega$

$$\begin{aligned} 6 - 5cos\omega &= a^2 + abe^{-i\omega} + abe^{i\omega} + b^2 \\ &= a^2 + b^2 + 2abcos\omega \ . \end{aligned}$$

Therefore we have the following system

$$\begin{cases} a^2 + b^2 = 6 \\ 2ab = -5 \ , \end{cases}$$

Thus

$$a = \frac{-5}{2b} \qquad and \qquad 4b^4 - 24b^2 + 25 = 0 \ .$$

Resolving this equation of the second degree we have

$$b = \pm\sqrt{\frac{6 \pm \sqrt{11}}{2}}$$

Therefore;

$$\text{when} \qquad b = \frac{1}{2} + \frac{\sqrt{11}}{2} \quad , \quad a = \frac{1}{2} - \frac{\sqrt{11}}{2}$$

in the same way

$$\text{when} \qquad b = \frac{1}{2} - \frac{\sqrt{11}}{2} \quad , \quad a = \frac{1}{2} + \frac{\sqrt{11}}{2}.$$

Thus

$$Q_2(e^{i\omega}) = \frac{1}{2}\left\{1 \pm \sqrt{11} + \left(1 \mp \sqrt{11}\right)e^{-i\omega}\right\}.$$

Now we can find the scaling sequence. We know that $P = H^2 R_2 = H^2 |Q_2|^2$, thus

$$\begin{aligned} P(e^{i\omega}) &= \left|\left(\frac{1 + e^{-i\omega} + e^{-i2\omega} + e^{-i3\omega}}{4}\right)^2\right|^2 |a + be^{i\omega}|^2 \\ &= \left(\frac{1 + e^{-i\omega} + e^{-i2\omega} + e^{-i3\omega}}{4}\right)^2 \left(\frac{1 + e^{i\omega} + e^{i2\omega} + e^{i3\omega}}{4}\right)^2 (a + be^{-i\omega})\overline{(a + be^{-i\omega})}. \end{aligned}$$

As $P(e^{i\omega}) = m_0(e^{i\omega})\overline{m_0(e^{i\omega})}$

$$\begin{aligned} m_0(e^{i\omega}) &= \left(\frac{1 + e^{-i\omega} + e^{-i2\omega} + e^{-i3\omega}}{4}\right)^2 (a + be^{i\omega}) \\ &= \frac{1}{16}\left[1 + 2e^{-i\omega} + 3e^{-i2\omega} + 4e^{-i3\omega} + 3e^{-i4\omega} + 2e^{-i5\omega} + e^{-i6\omega}\right](a + be^{-i\omega}) \end{aligned}$$

Thus $m_0(e^{i\omega}) = \frac{1}{4}\sum_{k=0}^{7} a_{0,k}e^{-ik\omega}$ and the coefficients $a_{0,k}$ are given by

$$a_{0,0} = \frac{a}{4} = \frac{1}{4}\left(\frac{1}{2} \pm \frac{\sqrt{11}}{2}\right) = \frac{1}{4}\left[\frac{1 \pm \sqrt{11}}{2}\right] = \left[\frac{1 \pm \sqrt{11}}{8}\right]$$

$$a_{0,1} = \frac{(2a+b)}{4} = \frac{1}{4}\left[2\left[\frac{1 \pm \sqrt{11}}{2}\right] + \left[\frac{1 \mp \sqrt{11}}{2}\right]\right] = \frac{3 \pm \sqrt{11}}{8}.$$

Proceeding in this way, we obtain that the scaling sequence for $M = 4$ and $N = 2$ is given by

$$\left\{a_{0,k}\right\} = \left\{\frac{1 \pm \sqrt{11}}{8}, \frac{3 \pm \sqrt{11}}{8}, \frac{5 \pm \sqrt{11}}{8}, \frac{7 \pm \sqrt{11}}{8}, \frac{7 \mp \sqrt{11}}{8}, \frac{5 \mp \sqrt{11}}{8}, \frac{3 \mp \sqrt{11}}{8}, \frac{1 \mp \sqrt{11}}{8}\right\}.$$

## 5. Construction of M-wavelet matrices

The objective of this section is to construct a M-wavelet matrix from a succession of scale with N vanishing moments. Remember that a scaling sequence satisfies (3.1) and ( 3.2) and an M-wavelet matrix must satisfy (2.1) and (2.2).

Before starting the construction we will give some notation and definitions. The M-wavelet matrix *A* will have M rows and K columns; it is convenient to add zeros to each row of *A* necessary for K = Mg for some integer g. We will say that g is the overlap of the M-wavelet matrix *A*.

A matrix $H = (h_{s,k})_{s=0}^{M-1}{}_{k=0}^{M-1}$ order $M \times M$ is said to be **Haar type** if

$$\sum_{k=0}^{M-1} h_{s,k}h_{s',k} = M\delta_{s,s'} \quad s, s' = 0, 1, ..., M-1 \tag{5.1}$$

and

$$h_{0,k} = 1 \quad \text{para todo} \quad k = 0, 1, ..., M-1 \tag{5.2}$$

Therefore, $H$ is a Haar type matrix if $\frac{1}{\sqrt{M}}H$ is orthogonal and the first row of $H$ is all ones.

Examples of Haar type matrices are the matrices of the discrete cosines transforms (DCT) that have been mentioned at the end of section 3 and the Hadamard's matrix

$$H_2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \; H_4 = \begin{bmatrix} H_2 & H_2 \\ -H_2 & H_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Given an M-wavelet matrix $A$ of order $M \times Mg$, $g \in \mathbf{Z}$, we write $A$ as $g$ matrices each of order $M \times M$, separating each M columns from the form

$$A = (A_0, A_1 ... A_{g-1})$$

**Definition 5.1.** *The matrix $H_0 = A_0 + A_1 + ... + A_{g-1}$ it's called* **matrix characteristic of Haar** *associated with A.*

**Lemma 5.1.** *If A is an M-wavelet matrix of order $M \times Mg$, $g \in \mathbf{Z}$, the matrix $H_0$ given in the definition (5.1) is a Haar type matrix.*

**Proof.** Let $H_0 = (h_{s,k})_{s=0}^{M-1}{}_{,k=0}^{M-1}$ be and $A = (a_{s,k})_{s=0}^{M-1}{}_{,k=0}^{Mg-1}$; then $h_{s,k} = \sum_{l=0}^{g-1} a_{s,k+lM}$ for $s = 0, 1, ..., M-1$. Thus

$$\begin{aligned}\sum_{k=0}^{M-1} h_{s,k}h_{s',k} &= \sum_{k=0}^{M-1}\Big(\sum_{l=0}^{g-1} a_{s,k+lM}\Big)\Big(\sum_{l'=0}^{g-1} a_{s',k+l'M}\Big)\\ &= \sum_{l'=0}^{g-1}\Big\{\sum_{k=0}^{M-1}\sum_{l=0}^{g-1} a_{s,k+lM}a_{s',k+l'M}\Big\}.\end{aligned}$$

For $l' = l$ in (2.1)

$$\sum_{k=0}^{M-1}\sum_{l=0}^{g-1} a_{s,k+lM}a_{s',k+l'M} = \sum_{k=0}^{Mg-1} a_{s,k}a_{s',k} = M\delta_{s,s'}$$

For $l' \neq l$ in (2.1)

$$\sum_{l'\neq l,\ l=0}^{g-1}\Big\{\sum_{k=0}^{M-1}\sum_{l=0}^{g-1} a_{s,k+lM}a_{s',k+l'M}\Big\} = \sum_{l'\neq l,\ l=0}^{g-1}\ \sum_{k=0}^{Mg-1} a_{s,k}a_{s',k+(l'-l)M} = 0$$

with this we have that

$$\sum_{k=0}^{M-1} h_{s,k}h_{s',k} = M\delta_{s,s'}$$

Now we need to proof (5.2). We just proved that $\sum_{k=0}^{M-1} h_{0,k}^2 = M$. Accordingly, the vector $v_0 = (h_{0,1}, ..., h_{0,M-1})$ is in the sphere of center 0 and radius $\sqrt{M}$ in $\mathbf{R}^M$. Also by (2.2),

$$\sum_{k=0}^{M-1} h_{0,k} = \sum_{k=0}^{M-1}\Big(\sum_{l=0}^{g-1} a_{0,k+lM}\Big) = \sum_{k=0}^{Mg-1} a_{0,k} = M$$

which implies that $v_0$ is also a point of the plane $\sum_{i=1}^{M} x_i = M$ en $\mathbf{R}^M$. Distance from the origin to this plane is reached in $(1, 1, ..., 1) \in \mathbf{R}^M$ and its value is $\sqrt{1+1+...+1} = \sqrt{M}$. Therefore, the sphere and plane considered are tangent to each other and the point of tangency is $(1, 1, ..., 1)$. This proof that $h_{0,k} = 1$ for all $k = 0, 1, ..., M-1$. ■

Given an M-wavelet matrix $A$ we call **polyphase matrix** associated with $A$ to the matrix

$$H(z) = A_0 + zA_1 + ... + z^{g-1}A_{g-1}.$$

Note that the elements of $H(z)$ are

$$h_{s,r}(z) = \sum_{l=0}^{g-1} a_{s,r+lM}z^l, \quad s, r = 0, 1, ..., M-1,$$

and $H(z)|_{z=1} = H_0$ (see definition 5.1).

**Lemma 5.2.** *The condition (2.1) of the M-wavelet matrix is equivalent to*

$$H(z)H^*(z) = MI \qquad if \quad |z| = 1.$$

**Proof.**

$$\begin{aligned} MI &= H(e^{i\omega})H^*(e^{i\omega}) = \left(\sum_{n=0}^{g-1} A_n e^{in\omega}\right)\left(\sum_{p=0}^{g-1} A_p^t e^{-ip\omega}\right) \\ &= \sum_{n=0}^{g-1}\sum_{p=0}^{g-1} A_n A_p^t e^{i(n-p)\omega} \\ &= \sum_{l=1}^{g-1}\left(\sum_{n=0}^{g-l-1} A_n A_{n+l}^t\right)e^{-il\omega} + \sum_{n=0}^{g-1} A_n A_n^t + \sum_{l=1}^{g-1}\left(\sum_{n=l}^{g-1} A_n A_{n-l}^t\right)e^{il\omega}. \end{aligned}$$

This implies that

$$\sum_{n=0}^{g-1} A_n A_n^t = MI \tag{5.3}$$

$$\sum_{n=0}^{g-l-1} A_n A_{n+l}^t = 0 \qquad \forall\ l = 1, 2, ..., g-1 \tag{5.4}$$

$$\sum_{n=l}^{g-1} A_n A_{n-l}^t = 0 \qquad \forall\ l = 1, 2, ..., g-1. \tag{5.5}$$

Note that (5.4) is equivalent to (5.5). Also (5.3) is equivalent to

$$\sum_{n=0}^{g-1}\sum_{k=0}^{M-1} a_{s,k+Mn}a_{s',k+Mn} = \sum_{k=0}^{Mg-1} a_{s,k}a_{s',k} = M\delta_{s,s'},$$

and on the other hand (5.4) is equivalent to

$$\sum_{n=0}^{g-l-1}\sum_{k=0}^{M-1} a_{s,k+Mn}a_{s',k+M(n-l)} = \sum_{k=0}^{M(g-l)-1} a_{s,k}a_{s',k+Ml} = 0 \qquad \forall\ l = 1, 2, ..., g-1$$

■

**Theorem 5.1.** *Let $a_0 = (a_{0,0}, ..., a_{0,gM-1})$ be a succession of scale with overlap $g \in \mathbf{Z}$. Let $H_0$ be an matrix of type Haar. Then, there is an M-wavelet matrix $A = (a_{s,k})_{s=0,\ k=0}^{M-1\ Mg-1}$ whose first row is $a_0$ and whose characteristic matrix of Haar is $H_0$ such that its polyphase matrix $H(z)$ can be written in the form*

$$H(z) = \left(\prod_{k=0}^{g-2}(I - v_k v_k^t + z v_k v_k^t)\right)H_0 \tag{5.6}$$

*with $v_k = (v_{k,1}, ..., v_{k,M})^t$ unit vectors in $\mathbf{R}^M$. Also the prime factors $I - v_k v_k^t + z v_k v_k^t \in \mathbf{M}_{M\times M}$ and M-wavelet matrix $A$ can be constructed explicitly from $a_0$ and $H_0$.*

**Proof.** We wish to obtain $v_k$ such that the relationship (5.6) holds. Multiplying by $H_0^{-1}$ to the right you get

$$H(z)H_0^{-1} = A_0H_0^{-1} + zA_1H_0^{-1} + ... + z^{g-1}A_{g-1}H_0^{-1} = \prod_{k=0}^{g-2}(I - v_k v_k^t + z v_k v_k^t).$$

Doing $B_k^0 = \frac{1}{M} A_k H_0^t$ we have that

$$H(z)H_0^{-1} = B_0^0 + zB_1^0 + ... + z^{g-1}B_{g-1}^0 \equiv \prod_{k=0}^{g-2}(I - v_k v_k^t + z v_k v_k^t). \tag{5.7}$$

The first row of each $B_k^0$ is known, since we know $H_0^{-1}$ and the first row of $A_k$; but the remaining $M-1$ rows of each matrix $B_k^0$ are indeterminate. We will denote by $\beta_k^0$ the first row of the matrix $B_k^0$ of order $M \times M$. If we write $\alpha_k$ for the subvectors of length $M$ of the scaling sequence $\{a_{0,k}\}$, then as $\{a_{0,k}\}$ is a scaling sequence and

$$\sum_{k=0}^{(g-l)M-1} a_{0,k}a_{0,k+Ml} = M\delta_{0,l},$$

we have that

$$\sum_{k=0}^{g-1-l} \alpha_{k+l}\alpha_k^t = M\delta_{0,l}. \tag{5.8}$$

On the other hand

$$\sum_{k=0}^{g-1} \alpha_k = (1, 1, ..., 1) \tag{5.9}$$

since $H_0 = A_0 + A_1 + ... + A_{g-1}$ is a Haar matrix (see Lemma 5.2).

We will show that

$$\sum_{k=0}^{g-1} \beta_k^0 = (1, 0, ..., 0) \tag{5.10}$$

and

$$\sum_{k=0}^{g-1-l} \beta_{k+l}^0 \beta_k^{0t} = \delta_{0,l} \tag{5.11}$$

To demonstrate (5.10) note that

$$\sum_{k=0}^{g-1}\beta_k^0 = \sum_{k=0}^{g-1}\alpha_k \frac{1}{M}H_0^t = \frac{1}{M}\Big(\sum_{k=0}^{g-1}\sum_{i=0}^{M-1} a_{0,i+kM}, \sum_{k=0}^{g-1}\sum_{i=0}^{M-1} a_{0,i+kM}h_{1,i}, ..., \sum_{k=0}^{g-1}\sum_{i=0}^{M-1} a_{0,i+kM}h_{M-1,i}\Big).$$

From (2.2) it follows

$$\frac{1}{M}\sum_{k=0}^{g-1}\sum_{i=0}^{M-1} a_{0,i+kM} = \frac{1}{M}\sum_{k=0}^{Mg-1} a_{0,k} = 1 \quad \text{para} \quad l = 1, 2, ..., M-1;$$

From (5.1) it follows

$$\frac{1}{M}\sum_{k=0}^{g-1}\sum_{i=0}^{M-1} a_{0,i+kM}h_{l,i} = \frac{1}{M}\sum_{i=0}^{M-1}\Big(\sum_{k=0}^{g-1} a_{0,i+kM}\Big)h_{l,i} = \frac{1}{M}\sum_{i=0}^{M-1} h_{0,i}h_{l,i} = 0.$$

To obtain (5.11) as $B_k^0 = \frac{1}{M}A_k H_0^t$ we have that $\beta_k^0 = \frac{1}{M}\alpha_k H_0^t$, then

$$\beta_k^{0t} = \frac{1}{M}H_0^{tt}\alpha_k^t = \frac{1}{M}H_0\alpha_k^t.$$

Thus

$$\beta^0_{k+l}\beta^{0t}_k = \frac{1}{M^2}\alpha_{k+l}H_0^t H_0 \alpha_k^t = \frac{1}{M^2}\alpha_{k+l} M I \alpha_k^t = \frac{1}{M}\alpha_{k+l}\alpha_k^t$$

therefore,

$$\sum_{k=0}^{g-1-l}\beta^0_{k+l}\beta^{0t}_k = \frac{1}{M}\sum_{k=0}^{g-1-l}\alpha_{k+l}\alpha_k^t = \frac{1}{M}M\delta_{0,l} = \delta_{0,l}.$$

As

$$B^0_{g-1} = v_0(v_0^t v_1)(v_1^t v_2)...(v_{g-3}^t v_{g-2})v_{g-2}^t$$

and $v_k^t v_{k+1} \in \mathbf{R}$ for all $k = 0, 1, ..., g-3$ we deduce that $B^0_{g-1} = \lambda v_0 v_{g-2}^t$ for some $\lambda \in \mathbf{R}$. Therefore, the rows of $B^0_{g-1}$ are proportional to $v_{g-2}^t$ and $B^0_{g-1}$ has rank 1

Also, if we write $v_0 = (v_0^0, ..., v_0^{M-1})^t$ you have that $\beta^0_{g-1} = \lambda v_0^0 v_{g-2}^t$. Since $v_{g-2}$ must be unitary, we have that

$$v_{g-2}^t = \frac{\beta^0_{g-1}}{||\beta^0_{g-1}||}. \tag{5.12}$$

Note that all rows of $B^0_{g-1}$ are multiples of the first row. The next step is to find $v_{g-3}$. It is easy to verify that

$$\left(I - v_{g-2}v_{g-2}^t + z v_{g-2}v_{g-2}^t\right)^{-1} = I - v_{g-2}v_{g-2}^t + z^{-1}v_{g-2}v_{g-2}^t$$

Multiplying in (5.7) by the inverse matrix of $\left(I - v_{g-2}v_{g-2}^t + z v_{g-2}v_{g-2}^t\right)$ we have that

$$\begin{aligned} H(z)H_0^{-1}(I - v_{g-2}v_{g-2}^t + z^{-1}v_{g-2}v_{g-2}^t) &= \prod_{k=0}^{g-3}(I - v_k v_k^t + z v_k v_k^t) \\ &\equiv B_0^1 + zB_1^1 + ... + z^{g-2}B^1_{g-2} \end{aligned} \tag{5.13}$$

As we know the matrix $H_0^{-1}(I - v_{g-2}v_{g-2}^t + z^{-1}v_{g-2}v_{g-2}^t)$ and the first row of $H(z)$, then we know each of the first rows $\beta_k^1$ of the $B_k^1$ for $k = 0, 1, ..., g-2$ then

$$\begin{aligned} B_0^1 + zB_1^1 + ... + z^{g-2}B^1_{g-2} &= \sum_{s=0}^{g-2} B_s^1 z^s \\ &= (B_0^0 + zB_1^0 + ... + z^{g-2}B^0_{g-2})(I - v_{g-2}v_{g-2}^t + z^{-1}v_{g-2}v_{g-2}^t) \\ &= \sum_{s=0}^{g-1} B_s^0(I - v_{g-2}v_{g-2}^t)z^s + \sum_{s=0}^{g-1} B_s^0 z^{s-1} v_{g-2}v_{g-2}^t. \end{aligned}$$

Making a change of variable, we have that

$$B_0^1 + zB_1^1 + ... + z^{g-2}B^1_{g-2} = \sum_{s=0}^{g-1} B_s^0(I - v_{g-2}v_{g-2}^t)z^s + \sum_{s=-1}^{g-2} B^0_{s+1} z^s v_{g-2}v_{g-2}^t;$$

Thus

$$B_0^0 v_{g-2} v_{g-2}^t = 0 \qquad \Rightarrow \qquad \beta_0^0 v_{g-2} v_{g-2}^t = 0 \tag{5.14}$$

$$\begin{aligned} & B_k^1 = B_k^0(I - v_{g-2}v_{g-2}^t) + B_{k+1}^0 v_{g-2} v_{g-2}^t \\ \Rightarrow \quad & \beta_k^1 = \beta_k^0(I - v_{g-2}v_{g-2}^t) + \beta_{k+1}^0 v_{g-2} v_{g-2}^t \quad , k = 0, ..., g-2 \end{aligned} \tag{5.15}$$

$$B_{g-1}^0(I - v_{g-2}v_{g-2}^t) = 0 \qquad \Rightarrow \qquad \beta_{g-1}^0(I - v_{g-2}v_{g-2}^t) = 0. \tag{5.16}$$

Therefore

$$\sum_{k=0}^{g-2-l} \beta_{k+l}^1 \beta_k^{1t} = \delta_{0,\,l} \quad , l = 0, 1, ..., g-2 \tag{5.17}$$

and

$$\sum_{k=0}^{g-2} \beta_k^1 = (1, 0, ..., 0). \tag{5.18}$$

To test (5.17) we use (5.15) getting

$$\begin{aligned} \sum_{k=0}^{g-2-l} \beta_{k+l}^1 \beta_k^{1t} &= \sum_{k=0}^{g-2-l} \left[\beta_{k+l}^0(I - v_{g-2}v_{g-2}^t) + \beta_{k+l+1}^0 v_{g-2}v_{g-2}^t\right]\left[\beta_k^0(I - v_{g-2}v_{g-2}^t) + \beta_{k+1}^0 v_{g-2}v_{g-2}^t\right]^t \\ &= \sum_{k=0}^{g-2-l} \beta_{k+l}^0(I - v_{g-2}v_{g-2}^t)(I - v_{g-2}v_{g-2}^t)\beta_k^{0t} + \sum_{k=0}^{g-2-l} \beta_{k+l}^0(I - v_{g-2}v_{g-2}^t)v_{g-2}v_{g-2}^t\beta_{k+1}^{0t} \\ &+ \sum_{k=0}^{g-2-l} \beta_{k+l+1}^0 v_{g-2}v_{g-2}^t(I - v_{g-2}v_{g-2}^t)\beta_k^{0t} + \sum_{k=0}^{g-2-l} \beta_{k+l+1}^0 v_{g-2}v_{g-2}^t v_{g-2}v_{g-2}^t\beta_{k+1}^{0t} \\ &= \sum_{k=0}^{g-2-l} \beta_{k+l}^0(I - v_{g-2}v_{g-2}^t)\beta_k^{0t} + \sum_{k=0}^{g-2-l} \beta_{k+1+l}^0 v_{g-2}v_{g-2}^t\beta_{k+1}^{0t}. \end{aligned}$$

By (5.16) we have that

$$\sum_{k=0}^{g-2-l} \beta_{k+l}^0(I - v_{g-2}v_{g-2}^t)\beta_k^{0t} = \sum_{k=0}^{g-1-l} \beta_{k+l}^0(I - v_{g-2}v_{g-2}^t)\beta_k^{0t}.$$

Making the change of variable $s = k + 1$ and applying (5.14) we have that

$$\sum_{k=0}^{g-2-l} \beta_{k+1+l}^0 v_{g-2}v_{g-2}^t\beta_{k+1}^{0t} \quad = \quad \sum_{s=0}^{g-1-l} \beta_{s+l}^0 v_{g-2}v_{g-2}^t\beta_s^{0t}.$$

Thus

$$\begin{aligned} \sum_{k=0}^{g-2-l} \beta_{k+l}^1 \beta_k^{1t} &= \sum_{k=0}^{g-1-l} \beta_{k+l}^0(I - v_{g-2}v_{g-2}^t)\beta_k^{0t} + \sum_{s=0}^{g-1-l} \beta_{s+l}^0 v_{g-2}v_{g-2}^t\beta_s^{0t} \\ &= \sum_{k=0}^{g-1-l} \beta_{k+l}^0 \beta_k^{0t} \\ &= \delta_{0,l} \qquad \text{by (5.11).} \end{aligned}$$

To test (5.18) we use (5.14), (5.15) and (5.16) getting

$$\begin{aligned}
\sum_{k=0}^{g-2}\beta_k^1 &= \sum_{k=0}^{g-2}\beta_k^0(I - v_{g-2}v_{g-2}^t) + \beta_{k+1}^0 v_{g-2}v_{g-2}^t \\
&= \sum_{k=0}^{g-1}\beta_k^0(I - v_{g-2}v_{g-2}^t) + \sum_{k=0}^{g-2}\beta_{k+1}^0 v_{g-2}v_{g-2}^t \\
&= \sum_{k=0}^{g-1}\beta_k^0 \\
&= (1,0,...,0) \qquad \text{by (5.10)}
\end{aligned}$$

with this

$$(v_0v_0^t)(v_1v_1^t)...(v_{g-3}v_{g-3}^t) = B_{g-2}^1,$$

since we want $v_{g-3}^t$ unit, we have that

$$v_{g-3}^t = \frac{\beta_{g-2}^t}{||\beta_{g-2}^1||}, \tag{5.19}$$

where $\beta_{g-2}^1$ is known.

We iterate this procedure to determine $v_{g-4},...,v_1$ until you reach the point where you have to find $v_0$ from

$$B_0^{g-2} + zB_1^{g-2} = I - v_0v_0^t + zv_0v_0^t. \tag{5.20}$$

So, we deduce that $B_1^{g-2} = v_0v_0^t$, thus

$$v_0^t = \frac{\beta_1^{g-2}}{||\beta_1^{g-2}||}.$$

To finish we must show that $H(e^{i\omega})H^t(e^{-i\omega}) = MI$, by lemma 5.2 we deduce that $A = (A_0, A_1, ..., A_{g-1})$ is an M-wavelet matrix, but

$$\begin{aligned}
H(e^{i\omega})H^t(e^{-i\omega}) &= \prod_{k=0}^{g-1}(I - v_kv_k^t + e^{i\omega}v_kv_k^t)H_0H_0^t(I - v_kv_k^t + e^{-i\omega}v_kv_k^t) \\
&= M\prod_{k=0}^{g-1}(I - v_kv_k^t + e^{i\omega}v_kv_k^t)(I - v_kv_k^t + e^{-i\omega}v_kv_k^t) \\
&= MI.
\end{aligned}$$

■

In Theorem 5.1 we have started with a scaling sequence $a_0 = (a_{0,0}, a_{0,1}, ..., a_{0,Mg-1})$, a Haar matrix and we have found prime factors $I - v_kv_k^t + zv_kv_k^t,\ k = 0,1,...,g-2$ such that (5.6) produces a matrix $H(z)$ satisfying $H(z)H^*(z) = MI$ when $|z| = 1$. Writing $H(z)$ as $A_0 + zA_1 + ... + z^{g-1}A_{g-1}$ the matrix

$$A = (A_0, A_1, ..., A_{g-1})$$

of order $M\times M(g-1)$ is an M-wavelet matrix (by Lemma 5.2) and its characteristic Haar matrix is $H(z)|_{z=1} = H_0$.

## 6. Examples of M-wavelets with N vanishing moments

Now we will use the methods developed previously for constructing M-wavelet matrix with N vanishing moments. In all the examples we will take $N = 2$ and we will contruct matrices $A$ with overlap $g = 2$.

Let $H_0$ be an matrix of Haar type of order $M \times M$. As $N = g = 2$ the polyphase matrix is

$$H(z) = A_0 + zA_1 = (I - v_0 v_0^t + z v_0 v_0^t) H_0 \tag{6.1}$$

where $v_0^t$ is a row vector of $R^M$. Let $\alpha_0$ , $\alpha_1$ be the two subvectors of length M of the scaling sequence. Of (5.12) and the definition of $B_0$ and $B_1$ we deduce that

$$v_0^t = \frac{\beta_1^0}{||\beta_1^0||} \qquad \text{y} \quad \beta_1^0 = \frac{1}{M}\alpha_1 H_0^t.$$

As

$$||\beta_1^0||_2^2 = \beta_1^0.(\beta_1^0)^t = \frac{1}{M^2}\alpha_1 H_0^t H_0 \alpha_1^t = \frac{1}{M}\alpha_1\alpha_1^t,$$

we have that

$$v_0 v_0^t = \frac{M}{\alpha_1\alpha_1^t}\frac{1}{M^2}H_0\alpha_1^t\alpha_1 H_0^t = \frac{1}{\alpha_1\alpha_1^t}\frac{1}{M}H_0\alpha_1^t\alpha_1 H_0^t.$$

Thus of (6.1)

$$A_1 = v_0 v_0^t H_0 = \frac{1}{\alpha_1\alpha_1^t}H_0\alpha_1^t\alpha_1 \tag{6.2}$$

and

$$A_0 = (I - v_0 v_0^t)H_0 = H_0 - A_1. \tag{6.3}$$

With the formulas (6.2) and (6.3) we will make the examples next.

**Example 6.1:** Let $M = 3$, $N = 2$ and $H_0 = DCT$ of order 3.

From Example 4.1 of Section 4 we take the scaling sequence

$$\left\{a_{0,k}\right\} = \left\{\frac{3+\sqrt{57}}{18}, \frac{9+\sqrt{57}}{18}, \frac{15+\sqrt{57}}{18}, \frac{15-\sqrt{57}}{18}, \frac{9-\sqrt{57}}{18}, \frac{3-\sqrt{57}}{18}\right\},$$

and we write

$$\alpha_0 = \left(\frac{3+\sqrt{57}}{18}, \frac{9+\sqrt{57}}{18}, \frac{15+\sqrt{57}}{18}\right), \;\; \alpha_1 = \left(\frac{15-\sqrt{57}}{18}, \frac{9-\sqrt{57}}{18}, \frac{3-\sqrt{57}}{18}\right).$$

The DCT matrix of order 3 is (see chapter 9 of [HW])

$$H_0 = \begin{bmatrix} 1 & 1 & 1 \\ \sqrt{2}cos\frac{\pi}{6} & 0 & -\sqrt{2}cos\frac{\pi}{6} \\ \sqrt{2}cos\frac{\pi}{3} & -\sqrt{2} & \sqrt{2}cos\frac{\pi}{3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \frac{\sqrt{3}}{\sqrt{2}} & 0 & -\frac{\sqrt{3}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\sqrt{2} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

From (6.2) it follows,

$$A_1 = \frac{1}{\alpha_1\alpha_1^t} H_0\alpha_1^t\alpha_1 \approx \begin{bmatrix} 0.408600 & 0.0759177 & -0.249532 \\ 1.40158 & 0.260411 & -0.855950 \\ 0.00889283 & 0.00165241 & -0.00543108 \end{bmatrix}.$$

And from (6.3) it follows,

$$A_0 = H_0 - A_1 \approx \begin{bmatrix} 0.591399 & 0.924082 & 1.24953 \\ -0.176838 & -0.260411 & -0.368794 \\ 0.698213 & -1.41586 & 0.712537 \end{bmatrix}.$$

Therefore, a 3-wavelet matrix with 2 vanishing moments is

$$A = \begin{bmatrix} 0.591399 & 0.924082 & 1.24953 & 0.408600 & 0.0759177 & -0.249532 \\ -0.176838 & -0.260411 & -0.368794 & 1.40158 & 0.260411 & -0.855950 \\ 0.698213 & -1.41586 & 0.712537 & 0.00889283 & 0.00165241 & -0.00543108 \end{bmatrix}.$$

**Example 6.2:** Let $M = 4$, $N = 2$ and $H_0 = DCT$ of order 4.
From Example 4.2 of Secction 4 we take the scaling sequence

$$\left\{a_{0,k}\right\} = \left\{\frac{1+\sqrt{11}}{8}, \frac{3+\sqrt{11}}{8}, \frac{5+\sqrt{11}}{8}, \frac{7+\sqrt{11}}{8}, \frac{7-\sqrt{11}}{8}, \frac{5-\sqrt{11}}{8}, \frac{3-\sqrt{11}}{8}, \frac{1-\sqrt{11}}{8}\right\},$$

and we write

$$\alpha_0 = \Big(\frac{1+\sqrt{11}}{8}, \frac{3+\sqrt{11}}{8}, \frac{5+\sqrt{11}}{8}, \frac{7+\sqrt{11}}{8}\Big), \ \ \alpha_1 = \Big(\frac{7-\sqrt{11}}{8}, \frac{5-\sqrt{11}}{8}, \frac{3-\sqrt{11}}{8}, \frac{1-\sqrt{11}}{8}\Big).$$

The DCT matrix of order 3 is (see chapter 9 of [HW])

$$H_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \sqrt{2}cos\frac{\pi}{8} & \sqrt{2}cos\frac{3\pi}{8} & -\sqrt{2}cos\frac{3\pi}{8} & -\sqrt{2}cos\frac{\pi}{8} \\ 1 & -1 & -1 & 1 \\ \sqrt{2}cos\frac{3\pi}{8} & -\sqrt{2}cos\frac{\pi}{8} & \sqrt{2}cos\frac{\pi}{8} & -\sqrt{2}cos\frac{3\pi}{8} \end{bmatrix}.$$

From (6.2) it follows,

$$A_1 = \frac{1}{\alpha_1\alpha_1^t} H_0\alpha_1^t\alpha_1 \approx \begin{bmatrix} 0.460421 & 0.210422 & -0.03957 & -0.289577 \\ 1.50275 & 0.686788 & -0.129173 & -0.945143 \\ 0 & 0 & 0 & 0 \\ 0.106788 & 0.0488104 & -0.00917824 & -0.0671682 \end{bmatrix}.$$

And from (6.3) it follows,

$$A_0 = H_0 - A_1 \approx \begin{bmatrix} 0.539578 & 0.789577 & 1.03957 & 1.28957 \\ -0.196190 & -0.145592 & -0.412018 & -0.361420 \\ 1 & -1 & -1 & 1 \\ 0.434407 & -1.35537 & 1.31574 & -0.474027 \end{bmatrix}.$$

Therefore, a 4-wavelet matrix with 2 vanishing moments is

$$\begin{bmatrix} 0.53957 & 0.78957 & 1.0395 & 1.2895 & 0.46042 & 0.21042 & -0.0395 & -0.28957 \\ -0.19619 & -0.14559 & -0.41201 & -0.36142 & 1.5027 & 0.68678 & -0.12917 & -0.94514 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0.43440 & -1.3553 & 1.3157 & -0.47402 & 0.10678 & 0.04881 & -0.00917 & -0.06716 \end{bmatrix}.$$

**Example 6.3:** Let $M = 4$, $N = 2$ and $H_0$ = Hadamard matrix of order 4.
With $\{a_{0,k}\}$, $\alpha_0$ and $\alpha_1$ as in example 6.2 we take as Haar's matrix a Hadamard's matrix of order 4

$$H_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

From (6.2) it follows,

$$A_1 = \frac{1}{\alpha_1\alpha_1^t} H_0\alpha_1^t\alpha_1 \approx \begin{bmatrix} 0.460421 & 0.210422 & -0.03957 & -0.28957 \\ -0.673746 & -0.307915 & -0.0579155 & 0.423746 \\ -1.34749 & -0.615831 & 0.115831 & 0.847493 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

And from (6.3) it follows,

$$A_0 = H_0 - A_1 = \begin{bmatrix} 0.539578 & 0.789577 & 1.03957 & 1.28957 \\ -0.326253 & 1.30791 & -1.05791 & 0.576253 \\ 0.341493 & -0.384168 & 0.884168 & 0.152506 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Therefore, other 4-wavelet matrix with 2 vanishing moments is

$$\begin{bmatrix} 0.539578 & 0.789577 & 1.03957 & 1.28957 & 0.460421 & 0.210422 & -0.03957 & -0.28957 \\ -0.326253 & 1.30791 & -1.05791 & 0.576253 & -0.326253 & 1.30791 & -1.05791 & 0.576253 \\ 0.341493 & -0.384168 & 0.884168 & 0.152506 & -1.34749 & -0.615831 & 0.115831 & 0.847493 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

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