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On some Hermite-Hadamard type inequalities for functions whose second derivative are convex generalized.

Desigualdades de tipo Hermite-Hadamard para funciones cuya segunda derivada es convexa generalizada.

[1]Yenny Carolina Rangel-Oliveros & [2]Miguel José Vivas-Cortez

[1] *ycrangel@puce.edu.ec* & [2]*mjvivas@puce.edu.ec*

[1, 2] *Pontificia Universidad Católica del Ecuador*
Facultad de Ciencias Naturales y Exactas
Escuela de Ciencias Físicas y Matemáticas, Sede Quito, Ecuador.

Abstract

In this paper, we establish some new results related to the left-hand of the Hermite-Hadamard type inequalities for the class of functions whose second derivatives are s-φ-convex functions.

Keywords:
$\varphi-$convex function, $s-\varphi-$convex function, $s-$convex function, Hermite-Hadamard type inequalities.

Resumen

En este artículo establecemos algunos nuevos resultados relacionados a desigualdades del tipo Hermite-Hadamard para funciones cuya segunda derivada es $s-\varphi-$convexa.

Palabras claves: funciones $\varphi-$convexas, funciones $s-\varphi-$convexas, funciones $s-$convexas, desigualdades del tipo Hermite-Hadamard.

1. Introducción

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. Note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave. For some results which generalize, improve and extend the inequalities (1) we refer the reader to the recent papers [6, 8, 11, 12, 18].

The convex functions play a significant role in many fields, for example in biological system, economy, optimization and so on [14, 25]. And many important inequalities are established for these class of functions. Also the evolution of the concept of convexity has had a great impact in the community of investigators. In recent years, for example, generalized concepts such as s-convexity (see[9]), h-convexity (see [26, 29]), m-convexity (see [6, 11]), MT- convexity (see[18]) and others, as well as combinations of these new concepts have been introduced.

The role of convex sets, convex functions and their generalizations are important in applied mathematics specially in nonlinear programming and optimization theory. For example in economics, convexity plays a fundamental role in equilibrium and duality theory. The convexity of sets and functions have been the object of many studies in recent years. But in many new problems encountered in applied mathematics the notion of convexity is not enough to reach favorite results and hence it is necessary to extend the notion of convexity to the new generalized notions. Recently, several extensions have been considered for the classical convex functions such that some of these new concepts are based on extension of the domain of a convex function (a convex set) to a generalized form and some of them are new definitions that there is no generalization on the domain but on the form of the definition. Some new generalized concepts in this point of view are pseudo-convex functions [19], quasi-convex functions [4], invex functions [15], preinvex functions and Hermite-Hadamard-Féjer Type inequalities for strongly (s, m)-convex functions with modulus C, in the second sense [8].

2. Preliminaries

Recall that a real-valued function f defined in a real interval J is said to be convex if for all $x, y \in J$ and for any $t \in [0, 1]$ the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{2}$$

holds. If inequality (2) is strict when we say that f is strictly convex, and if inequality (2) is reversed the function f is said to be concave. In [16] Hudzik H. and Maligranda L. introduced the following generalized concept.

Definition 2.1. *Let $0 < s \leq 1$. The function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called a s-convex function in first sense if*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t^s)f(y) \tag{3}$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

It can be easily see that for $s = 1$ s-convexity reduces to the ordinary convexity of functions defined on $[0, \infty)$. Besides in [13], Gordji M. E., Delavar M. R., De la Sen M. introduced the definition $\varphi-$ convex function, where $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a bifunction except for special case.

Definition 2.2. *A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called convex with respect to φ (briefly $\varphi-$ convex), if*

$$f(tx + (1-t)y) \leq f(y) + t\varphi(f(x), f(y)), \tag{4}$$

for all $x, y \in I$ and $t \in [0, 1]$.

In fact above definition geometrically says that if a function is $\varphi-$ convex on I, then it's graph between any $x, y \in I$ is on or under the path starting from $(y, f(y))$ and ending at $(x, f(y) + \varphi(f(x), f(y)))$. If $f(x)$ should be the end point of the path for every $x, y \in I$, then we have $\varphi(x, y) = x - y$ and the function reduces to a convex one. Note that by taking $x = y$ in (4) we get $t\varphi(f(x), f(x)) \geq 0$ for any $x \in I$ and $t \in [0, 1]$ which implies that

$$\varphi(f(x), f(x)) \geq 0$$

for any $x \in I$. Also if we take $t = 1$ in (4) we get

$$f(x) - f(y) \leq \varphi(f(x), f(y))$$

for any $x, y \in I$. If $f : I \to \mathbb{R}$ is a convex function and $\varphi : I \times I \to \mathbb{R}$ is an arbitrary bifunction that satisfies

$$\varphi(x, y) \geq x - y$$

for any $x, y \in I$, then

$$f(tx + (1 - t)y) \leq f(y) + t[f(x) - f(y)] \leq f(y) + t\varphi(f(x), f(y))$$

showing that f is $\varphi-$convex.

In [32] Vivas-Cortez and Rangel-Oliveros introduced the following definition $s - \varphi$-convex functions as a generalization of $s-$convex functions in first sense.

Definition 2.3. *Let $0 < s \leq 1$. A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called $s - \varphi-$convex with respect to bifunction $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (briefly $\varphi-$convex), if*

$$f(tx + (1 - t)y) \leq f(y) + t^s\varphi(f(x), f(y)) \tag{5}$$

for all $x, y \in I$ and $t \in [0, 1]$.

In [32] we can see the following examples of functions $s - \varphi-$convex.

Example 2.4. *Let $f(x) = x^2$, then f is convex and $\frac{1}{2}$-φ- convex with $\varphi(u, v) = 2u + v$.*

Example 2.5. *Let $f(x) = x^n$ and $0 < s \leq 1$, then f is convex and s-φ-convex with*

$$\varphi(u, v) = \sum_{k=1}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) v^{1-\frac{k}{n}} \left(u^{\frac{1}{n}} - v^{\frac{1}{n}}\right)^n .$$

3. Main results

To prove our main results, we consider the following Lemma given by Ozdemir et al. in [23]

Lemma 3.1. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$. If $f'' \in L[a, b]$, then following equality holds*

$$\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) = \frac{(b-a)^2}{16}\left[\int_0^1 t^2 f''\left(t\frac{a+b}{2} + (1-t)a\right)dt + \int_0^1 (1-t)^2 f''\left(tb + (1-t)\frac{a+b}{2}\right)dt\right] \tag{6}$$

Theorem 3.2. *Suppose that $f : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on I° such that $f'' \in L([a,b])$, where $a,b \in I$ with $a < b$. If $|f''|$ is s-φ-convex on $[a,b]$, for some $s \in (0,1]$ then following inequalities hold*

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{(b-a)^2}{16}\left[\frac{1}{3}|f''(a)|+\frac{1}{s+3}\varphi\left(\left|f''\left(\frac{a+b}{2}\right)\right|,|f''(a)|\right)\right]\\ &+\frac{(b-a)^2}{16}\left[\frac{1}{3}\left|f''\left(\frac{a+b}{2}\right)\right|+\frac{2}{(s+1)(s+2)(s+3)}\varphi\left(|f''(b)|,\left|f''\left(\frac{a+b}{2}\right)\right|\right)\right]\end{aligned}$$

Proof. Taking modulus on both sides of (6), we have

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{16}\int_0^1 t^2\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt+\frac{(b-a)^2}{16}\int_0^1(1-t)^2\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt \quad (7)$$

Since $|f''|$ is s-φ-convex, we get

$$\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right| \leq |f''(a)|+t^s\varphi\left(\left|f''\left(\frac{a+b}{2}\right)\right|,|f''(a)|\right) \quad (8)$$

and

$$\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right| \leq \left|f''\left(\frac{a+b}{2}\right)\right|+t^s\varphi\left(|f''(b)|,\left|f''\left(\frac{a+b}{2}\right)\right|\right). \quad (9)$$

If we substitute the inequalities (8) and (9) in (7), then we obtain

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{(b-a)^2}{16}\int_0^1 t^2\left[|f''(a)|+t^s\varphi\left(\left|f''\left(\frac{a+b}{2}\right)\right|,|f''(a)|\right)\right]dt\\ &+\frac{(b-a)^2}{16}\int_0^1(1-t)^2\left[\left|f''\left(\frac{a+b}{2}\right)\right|+t^s\varphi\left(|f''(b)|,\left|f''\left(\frac{a+b}{2}\right)\right|\right)\right]dt \quad (10)\end{aligned}$$

Using the facts that

$$\int_0^1 t^{s+2}dt=\int_0^1(1-t)^{s+2}dt=\frac{1}{s+3}$$

and

$$\int_0^1 t^2\,(1-t)^s dt=\int_0^1 t^s(1-t)^2dt=\frac{2}{(s+1)(s+2)(s+3)}$$

in (10), we obtain

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{(b-a)^2}{16}\left[\frac{1}{3}|f''(a)|+\frac{1}{s+3}\varphi\left(\left|f''\left(\frac{a+b}{2}\right)\right|,|f''(a)|\right)\right]\\ &+\frac{(b-a)^2}{16}\left[\frac{1}{3}\left|f''\left(\frac{a+b}{2}\right)\right|+\frac{2}{(s+1)(s+2)(s+3)}\varphi\left(|f''(b)|,\left|f''\left(\frac{a+b}{2}\right)\right|\right)\right]\end{aligned}$$

thus the proof is completed. ■

Corollary 3.3. *In Theorem 3.2 if we choose $\varphi(x,y) = x - y$, we have the following inequality*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{16(s+3)}\left[\frac{s}{3}|f''(a)| + \frac{2}{(s+1)(s+2)}|f''(b)| + \frac{(s+1)(s+2)(s+6)-6}{3(s+1)(s+2)}\left|f''\left(\frac{a+b}{2}\right)\right|\right],$$

for s-convex functions in the first sense.

Corollary 3.4. *In Corollary 3.3 if we choose $s = 1$, we have the following inequality*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{64}\left[\frac{1}{3}|f''(a)| + \frac{1}{3}|f''(b)| + 2\left|f''\left(\frac{a+b}{2}\right)\right|\right],$$

for convex functions.

The next theorem gives a new upper bound of the left Hadamard inequality for s-φ-convex mappings.

Theorem 3.5. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on I° such that $f^{''} \in L([a,b])$,where $a, b \in I$ with $a < b$. If $|f^{''}|^q$ is s-φ-convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality holds:*

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(|f''(a)|^q + \frac{1}{s+1}\varphi\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q, |f''(a)|^q\right)\right)^{\frac{1}{q}} \\ &+ \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q + \frac{1}{s+1}\varphi\left(|f''(b)|^q, \left|f''\left(\frac{a+b}{2}\right)\right|^q\right)\right)^{\frac{1}{q}},\end{aligned}$$

where $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 3.1 and using Holder's inequality, we have

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(b-a)^2}{16}\int_0^1 t^2\left|f''\left(t\frac{a+b}{2} + (1-t)a\right)\right|dt + \frac{(b-a)^2}{16}\int_0^1 (1-t)^2\left|f''\left(tb + (1-t)\frac{a+b}{2}\right)\right|dt \\ &\leq \frac{(b-a)^2}{16}\left[\left(\int_0^1 |t|^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(t\frac{a+b}{2} + (1-t)a\right)\right|^q dt\right)^{\frac{1}{q}} + \left(\int_0^1 |1-t|^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(tb + (1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right]. \quad (11)\end{aligned}$$

Since $|f^{''}|^q$ is s-φ-convex, we have

$$\int_0^1\left|f''\left(t\frac{a+b}{2} + (1-t)a\right)\right|^q dt \leq |f''(a)|^q + \frac{1}{s+1}\varphi\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q, |f''(a)|^q\right) \quad (12)$$

and

$$\int_0^1\left|f''\left(tb + (1-t)\frac{a+b}{2}\right)\right|^q dt \leq \left|f''\left(\frac{a+b}{2}\right)\right|^q + \frac{1}{s+1}\varphi\left(|f''(b)|^q, \left|f''\left(\frac{a+b}{2}\right)\right|^q\right). \quad (13)$$

By a simple computation, we have

$$\int_0^1 t^{2p}dt = \int_0^1 (1-t)^{2p}dt = \frac{1}{2p+1} \quad (14)$$

If we put (12)-(14) in (11), we obtain

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| &\leq \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(|f''(a)|^q+\frac{1}{s+1}\varphi\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q,|f''(a)|^q\right)\right)^{\frac{1}{q}}\\ &+ \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{s+1}\varphi\left(|f''(b)|^q,\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)\right)^{\frac{1}{q}}\end{aligned}$$

which completes the proof. ■

Corollary 3.6. *In Theorem 3.5 if we choose $\varphi(x,y)=x-y$, we have the following inequality*

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{s}{s+1}|f''(a)|^q+\frac{1}{s+1}\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\\ &+ \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{s}{s+1}\left|f''\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{s+1}|f''(b)|^q\right)^{\frac{1}{q}},\end{aligned}$$

for s-convex functions in the first sense.

Corollary 3.7. *In Corollary 3.6 if we choose $s=1$, we have the following inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left[\left(\frac{1}{2}|f''(a)|^q+\frac{1}{2}\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\frac{1}{2}\left|f''\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{2}|f''(b)|^q\right)^{\frac{1}{q}}\right],$$

for convex functions.

Theorem 3.8. *Let $f: I\subset[0,\infty)\to\mathbb{R}$ be a twice differentiable mapping on I° such that $f''\in L([a,b])$, where $a,b\in I$ with $a<b$. If $|f''|^q$ is s-φ-convex on $[a,b]$, for some fixed $s\in(0,1]$, and $q\geq 1$, then the following inequality holds*

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{\frac{1}{p}}\left(\frac{1}{3}|f''(a)|^q+\frac{1}{s+3}\varphi\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q,|f''(a)|^q\right)\right)^{\frac{1}{q}}\\ &+ \frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{\frac{1}{p}}\left(\frac{1}{3}\left|f''\left(\frac{a+b}{2}\right)\right|^q+\frac{2}{(s+1)(s+2)(s+3)}\varphi\left(|f''(b)|^q,\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)\right)^{\frac{1}{q}}.\end{aligned}$$

Proof. From Lemma 3.1 and using the power mean inequality, we have

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(b-a)^2}{16}\left[\int_0^1 t^2\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt+\int_0^1(1-t)^2\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right]$$

$$\leq \frac{(b-a)^2}{16}\left[\left(\int_0^1 t^2dt\right)^{\frac{1}{p}}\left(\int_0^1 t^2\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1(1-t)^2dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)^2\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right].$$

Since $|f''|^q$ is s-φ-convex, by a simple computations, we obtain

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| &\leq \frac{(b-a)^2}{16}\left(\int_0^1 t^2dt\right)^{\frac{1}{p}}\left(\int_0^1 t^2\left[|f''(a)|^q+t^s\varphi\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q,|f''(a)|^q\right)\right]dt\right)^{\frac{1}{q}}\\ &+ \frac{(b-a)^2}{16}\left(\int_0^1(1-t)^2dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)^2\left[\left|f''(\frac{a+b}{2})\right|^q+t^s\varphi\left(|f''(b)|^q,\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)\right]dt\right)^{\frac{1}{q}}\\ &= \frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{\frac{1}{p}}\left(\frac{1}{3}|f''(a)|^q+\frac{1}{s+3}\varphi\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q,|f''(a)|^q\right)\right)^{\frac{1}{q}}\\ &+ \frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{\frac{1}{p}}\left(\frac{1}{3}\left|f''\left(\frac{a+b}{2}\right)\right|^q+\frac{2}{(s+1)(s+2)(s+3)}\varphi\left(|f''(b)|^q,\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)\right)^{\frac{1}{q}}.\end{aligned}$$

This completes the proof. ■

Corollary 3.9. *In Theorem 3.8 if we choose $\varphi(x,y)=x-y$, we have the following inequality*

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{\frac{1}{p}}\left(\frac{s}{3(s+3)}|f''(a)|^q+\frac{1}{s+3}\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\\ &+ \frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{\frac{1}{p}}\left(\frac{(s+1)(s+2)(s+3)-6}{3(s+1)(s+2)(s+3)}\left|f''\left(\frac{a+b}{2}\right)\right|^q+\frac{2}{(s+1)(s+2)(s+3)}|f''(b)|^q\right)^{\frac{1}{q}},\end{aligned}$$

for s-convex functions in the first sense.

Corollary 3.10. *In Corollary 3.9 if we choose $s=1$, we have the following inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{\frac{1}{p}}\left[\left(\frac{1}{12}|f''(a)|^q+\frac{1}{4}\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\frac{1}{4}\left|f''\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{12}|f''(b)|^q\right)^{\frac{1}{q}}\right],$$

for convex functions.

4. Applications for some particular mappings

In this section we give some applications for the special case where the function $\varphi(f(x),f(y))=f(x)-f(y)$, in this case we have that f is s-convex in the first sense.

Example 4.1. *Let $s\in(0,1)$ and $p,q,r\in\mathbb{R}$, we define the function $f:[0,+\infty)\to\mathbb{R}$ as*

$$f(t)=\begin{cases} p & \text{if} \quad t=0\\ qt^s+r & \text{if} \quad t>0\end{cases}$$

we have that if $q\geq 0$ and $r\leq p$, then f is s-convex in the first sense (see [16]). If we do $\varphi(f(x),f(y))=f(x)-f(y)$, then f is s-φ-convex, but is not φ-convex because f is not convex.

Example 4.2. *In the previous example if $s=\frac{1}{2}$, $p=1$, $q=2$ and $r=1$ we have that $f:[0,+\infty)\to\mathbb{R}$,*

$$f(t)=\begin{cases} 1 & \text{if} \quad t=0\\ 2t^{\frac{1}{2}}+1 & \text{if} \quad t>0\end{cases},$$

is $\frac{1}{2}$-φ-convex, then if we define $g:[0,+\infty)\to\mathbb{R}$, $g(t)=\frac{8}{15}t^{\frac{5}{2}}+\frac{t^2}{2}$, then $g''(t)$ is $\frac{1}{2}$-φ-convex in $[0,+\infty)$ with $\varphi(f(x),f(y))=f(x)-f(y)$. Using Theorem 3.2, we get

$$\left|\frac{8}{60\sqrt{2}}(a+b)^{\frac{5}{2}}+\frac{1}{8}(a+b)^2-\frac{16}{105(b-a)}(b^{\frac{7}{2}}-a^{\frac{7}{2}})-\frac{1}{6(b-a)}(b^3-a^3)\right| \leq \frac{(b-a)^2}{16}\left[\frac{2}{21}a^{\frac{1}{2}}+\frac{14}{15\sqrt{2}}(a+b)^{\frac{1}{2}}-\frac{32}{105}b^{\frac{1}{2}}+\frac{2}{3}\right].$$

Example 4.3. *If we define $g(t)=\frac{t^4}{12}$ we have that $g''(t)$ is $\frac{1}{2}$-φ- convex and by Theorem 3.2 we have that*

$$\left|\frac{(a+b)^4}{192}-\frac{b^4+b^3a+b^2a^2+ba^3+a^4}{60}\right| \leq \frac{(b-a)^2}{16}\left[\frac{59}{210}a^2+\frac{7ab}{15}+\frac{27}{70}b^2\right]$$

Remark 1. In particular if we choose $a=x$ and $b=y$, we get a graphic representation of the inequality of the Example 4.3 (see Figure 1).

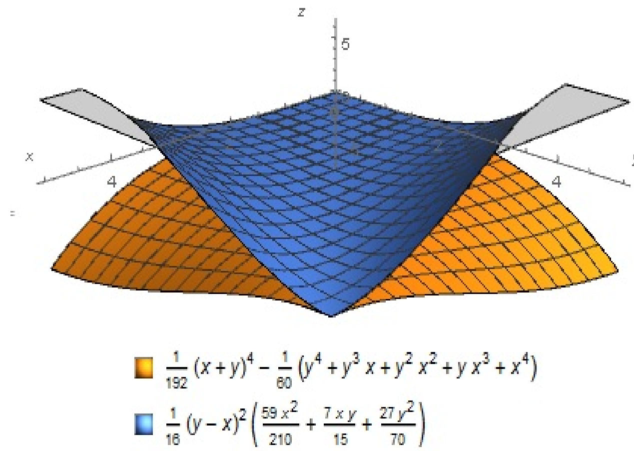

Figura 1.

Example 4.4. *If we define* $g(t) = \frac{36}{91}\sqrt[3]{2}\, t^{\frac{13}{6}}$ *we have that* $|g''(t)|^3$ *is* $\frac{1}{2}$*-*φ*-convex with* $\varphi(u,v) = u - v$ *and by Theorem 3.5, for* $a, b \in [0, +\infty)$ *with* $a < b$ *and* $x \in [a, b]$*, we have*

$$\left|\frac{36\sqrt[3]{2}}{91}\left(\frac{a+b}{2}\right)^{\frac{13}{6}} - \frac{216}{1729(b-a)}(b^{\frac{19}{6}} - a^{\frac{19}{6}})\right| \leq \frac{(b-a)^2}{128}\left[\left(\frac{10}{3}a^{\frac{1}{2}} - \frac{4}{3}\left(\frac{a+b}{2}\right)^{\frac{1}{2}}\right)^{\frac{1}{3}} + \left(\frac{10}{3}\left(\frac{a+b}{2}\right)^{\frac{1}{2}} - \frac{4}{3}b^{\frac{1}{2}}\right)^{\frac{1}{3}}\right]$$

5. Applications to special means

We now consider the means for arbitrary real numbers α, β $(\alpha \neq \beta)$. We take:

(1) Arithmetic mean:

$$A(\alpha,\beta) = \frac{\alpha+\beta}{2}, \quad \alpha,\beta \in \mathbb{R}^+;$$

(2) Generalized log-mean:

$$L_n(\alpha,\beta) = \left[\frac{\beta^{n+1} - \alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}}, \quad n \in \mathbb{Z} - \{-1, 0\}, \alpha, \beta \in \mathbb{R}^+.$$

Proposition 5.1. *Let* $0 < a < b$ *and* $s \in (0,1)$*. Then we have*

$$|A^s(a,b) - L_s^s(a,b)| \leq \frac{s(s-1)(b-a)^2}{16(s+3)}\left[\frac{s}{3}a^{s-2} + \frac{2}{(s+1)(s+2)}b^{s-2} + \frac{(s+1)(s+2)(s+6)-6}{3(s+1)(s+2)}\left(\frac{a+b}{2}\right)^{s-2}\right].$$

Proof. The assertion follows from Corollary 3.3 applied to the s-φ-convex function in the first sense $f : [0,1] \to [0,1]$, $f(x) = x^s$ with $\varphi(x,y) = x - y$. ■

Proposition 5.2. *Let* $0 < a < b$ *and* $s \in (0,1)$. *Then we have*

$$\begin{aligned}|A^s(a,b) - L_s^s(a,b)| &\leq \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(s^2(s-1)a^{q(s-2)} + s(s-1)\left(\frac{a+b}{2}\right)^{q(s-2)}\right)^{\frac{1}{q}}\\ &+ \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(s(s-1)\left(\frac{a+b}{2}\right)^{q(s-2)} + s(s-1)b^{q(s-2)}\right)^{\frac{1}{q}}.\end{aligned}$$

Proof. The assertion follows from Corollary 3.6 applied to the s-φ-convex function in the first sense $f : [0,1] \to [0,1]$, $f(x) = x^s$ with $\varphi(x,y) = x - y$. ■

Proposition 5.3. *Let* $0 < a < b$ *and* $s \in (0,1)$. *Then we have*

$$\begin{aligned}|A^s(a,b) - L_s^s(a,b)| &\leq \frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{\frac{1}{p}}\left(\frac{s^2(s-1)}{3(s+3)}a^{q(s-2)} + \frac{s(s-1)}{s+3}\left(\frac{a+b}{2}\right)^{q(s-2)}\right)^{\frac{1}{q}}\\ &+ \frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{\frac{1}{p}}\left(\frac{[(s+1)(s+2)(s+3)-6]s(s-1)}{3(s+1)(s+2)(s+3)}\left(\frac{a+b}{2}\right)^{q(s-2)} + \frac{2s(s-1)}{(s+1)(s+2)(s+3)}b^{q(s-2)}\right)^{\frac{1}{q}}.\end{aligned}$$

Proof. The assertion follows from Corollary 3.9 applied to the s-φ-convex function in the first sense $f : [0,1] \to [0,1]$, $f(x) = x^s$ with $\varphi(x,y) = x - y$. ■

6. Conclusions

In this paper we have established Hermite-Hadamard type inequalities given by Erdem-Ogunmez-Budak in [12] for the case s-φ-convex functions. We expect that the ideas and techniques used in this paper may inspire interested readers to explore some new applications of these newly introduced explore some new applications of these newly introduced functions in various fields of pure and applied sciences.

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