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OTHER ALGEBRAIC SOLUTIONS TO THE THIRD AND FOURTH GRADE POLYMONIC EQUATIONS

OTRAS SOLUCIONES ALGEBRAICAS A LAS ECUACIONES POLINOMICAS DE TERCER Y CUARTO GRADO

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Abstract

In this manuscript is shown first another reasoning to arrive at the construction of the Cardano‘s formula; with this same reasoning the determination of another formula for the solution of the equations of third degree and later two solutions for the equations of fourth degree is obtained; Finally, another solution is given to the fourth degree equation, treating it as an almost-symmetric equation, subjecting it to a transformation.

Keywords: Algebraic solution; polynomial equation; Cardano‘s formula; resolvent auxiliary equation; Taylor’s formula; almost-symmetric equation.

Resumen

En el presente manuscrito se muestra primeramente otro razonamiento para llegar a la construcción de la fórmula de Cardano; con este mismo razonamiento se logra la determinación de otra fórmula para la solución de las ecuaciones de tercer grado y posteriormente dos soluciones para las ecuaciones de cuarto grado; por último se da otra solución a la ecuación de cuarto grado tratándola como una ecuación casi-simétrica, sometiéndola a una trasformación.

Palabras claves: Solución algebraica; ecuación polinómica; formula de Cardano; ecuación auxiliar resolvente; formula de Taylor; ecuación casi-simétrica.

1. Introduction

Since the sixteenth century when the discovery of algebraic methods began to solve polynomial equations of the third and fourth degree, the mathematical community marveled at these formulations and their ingenious steps; At present there are many solutions and much faster approach methods to find the roots of the equations, however in this work we present new solutions to these equations that show beautifully ingenious methods that also allow to find these roots.

1. DETERMINATION OF THE CARDANO‘S FORMULA

Then a different reasoning is explained, and it will allow us to deduce Cardano's formula for the solution of the reduced third degree polynomial equation of the form:

$$X^3 + PX + Q = 0 \tag{1}$$

Consider the second degree equation: $Y^2 + C_0Y + C_1 = 0$
It‘s roots are a *and* b, so that: $a + b = -C_0$ $ab = C_1$
Now you can do the transformations with these coefficients, such that:

$$a^3 + b^3 = -C_0^3 + 3C_1C_0 = Q$$

$$C_0^3 - 3C_1C_0 + Q = 0$$

This last equation corresponds to equation (1) where:

$$C_0 = X$$

$$P = -3C_1 = -3ab$$

So that: $ab = -\frac{P}{3}$

$$a^3b^3 = -\frac{P^3}{27}$$

This allows to build the second degree resolvent auxiliary equation:

$$Z^2 - QZ - \frac{P^3}{27} = 0$$

And it‘s two roots a^3 *and* b^3 are:

$$a^3 = \frac{Q}{2} + \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}$$

$$b^3 = \frac{Q}{2} - \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}$$

And so $X = C_0 = a + b$
Then: $X = \sqrt[3]{\frac{Q}{2} + \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}} + \sqrt[3]{\frac{Q}{2} - \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}}$

1. **ANOTHER SOLUTION TO THE CUBIC EQUATION**

Using the same reasoning above, with respect to the second degree equation with roots a and b, it is also:

$$a^6 + b^6 = C_0^6 - 6C_1C_0^4 + 9C_1^2C_0^2 - 2C_1^3 = W$$

Compare this equation with: $X^3 + TX^2 + UX + Y = 0$
with: $C_0^2 = X$

$$T = -6C_1 \rightarrow C_1^2 = \frac{T^2}{36} \qquad \textbf{(2)}$$

$$U = 9C_1^2 \rightarrow C_1^2 = \frac{U}{9} \qquad \textbf{(3)}$$

$$Y = V - W$$

$$V = -2C_1^3$$

Now the identities (**??**) and (**??**) say that: $\frac{T^2}{36} = \frac{U}{9}$
Now for any third-degree equation to have this ratio in its coefficients, it must undergo a transformation using Taylor's formula:
$f(x+h) = (x+h)^n + \frac{f^{(n-1)}(h)(x+h)^{n-1}}{(n-1)!} + \frac{f^{(n-2)}(h)(x+h)^{n-2}}{(n-2)!} + \ldots\ldots\ldots + f^{(1)}(h)(x+h) + f(h)$ **(4)**
That in the case of cubic equations, it is:

$$f(x+h) = (x+h)^3 + \frac{f^{(2)}(h)(x+h)^2}{(2)!} + f^{(1)}(h)(x+h) + f(h) \quad \textbf{(5)}$$

Being $f^{(n)}(h)$ the nth derivative of $f(h)$.
where:

$$\frac{f^{(2)}(h)}{(2)!} = 3h + T = T_2$$

$$f^{(1)}(h) = 3h^2 + 2Th + U = U_2$$

And since the proportion is needed: $\frac{T_2^2}{36} = \frac{U_2}{9}$
Then: $\frac{(3h+T)^2}{36} = \frac{3h^2+2Th+U}{9}$
This last expression becomes a second degree equation: $h^2 + \frac{5}{3}h + \frac{4U-T^2}{3} = 0$
And it's roots are: $h = -\frac{5}{6} \pm \sqrt{\left(\frac{5}{6}\right)^2 + \frac{T^2-4U}{3}}$
Choosing a value of h, it is replaced in the formula (**??**) to obtain the cubic equation transformed with the proportion in the coefficients: $\frac{T^2}{36} = \frac{U}{9}$
Now with this proportion in the equation: $X^3 + TX^2 + UX + Y = 0$ (**??**)
it is observed that:

$$C_1 = -\frac{T}{6} = \frac{\sqrt{U}}{3}$$

$$W = \frac{T^3}{108} - Y = -\frac{2\left(\sqrt{U}\right)^3}{27} - Y$$

And as: $W = a^6 + b^6$ and ${C_1}^6 = a^6 b^6$
Represent the coefficients of the quadratic auxiliary solvent equation:

$$Y^2 - WY + {C_1}^6 = 0$$

And it's roots are:

$$a^6 = \frac{W}{2} + \sqrt{\frac{W^2}{4} - {C_1}^6}$$

$$b^6 = \frac{W}{2} - \sqrt{\frac{W^2}{4} - {C_1}^6}$$

And as in the equation (**??**)**:** $X = C_0^2 = (a+b)^2$
Then:

$$X = \left(\sqrt[6]{\frac{W}{2} + \sqrt{\frac{W^2}{4} - {C_1}^6}} + \sqrt[6]{\frac{W}{2} - \sqrt{\frac{W^2}{4} - {C_1}^6}}\right)^2$$

1. FIRST SOLUTION TO THE FOURTH GRADE EQUATION

Consider the third degree equation:

$$Y^3 + C_0Y^2 + C_1Y + C_2 = 0$$

It's roots are $a,\ b\ and\ c$, so that:

$$a^2 + b^2 + c^2 = C_0^2 + 2C_1 = L \tag{2}$$

$$a^2b^2 + a^2c^2 + b^2c^2 = C_1^2 + 2C_2C_0 = M \tag{3}$$

$$a^2b^2c^2 = C_2^2 = N$$

From (3) we have that: $C_1 = \sqrt{M + 2\sqrt{N}C_0}$
And replacing this identity in (2) and eliminating the root by squaring, so:

$$C_0^2 = L + 2\sqrt{M + 2\sqrt{N}C_0} \rightarrow C_0^4 - 2LC_0^2 + L^2 = 4M + 8\sqrt{N}C_0 \rightarrow$$

$$C_0^4 - 2LC_0^2 - 8\sqrt{N}C_0 + (L^2 - 4M) = 0$$

This last equation resembles a reduced fourth degree equation in its third degree term, in the form:

$$X^4 + PX^2 + QX + R = 0 \tag{4}$$

where:

$$X = C_0$$

$$P = -2L$$

$$Q = -8\sqrt{N}$$

$$R = L^2 - 4M$$

Recall that to eliminate the third degree term in the equation:

$$Z^4 + AZ^3 + BZ^2 + CZ + D = 0$$

$Z = X - \frac{A}{4}$ is replaced, so that it is of the form of (4).
Now it is so: $L = -\frac{P}{2}\ N = \frac{Q^2}{64}\ M = \frac{P^2-4R}{16}$
These are the coefficients of the cubic auxiliary solvent equation:

$$Y_1^3 - LY_1^2 + MY_1 - N = 0$$

And it's roots are $a^2, b^2\ and\ c^2$.
So from equation (4), $X = C_0 = \sqrt{a^2} + \sqrt{b^2} + \sqrt{c^2}$

1. SECOND SOLUTION TO THE FOURTH GRADE EQUATION

From the equation (3) we have that: $C_0 = \frac{C_1^2 - M}{2\sqrt{N}}$
And replacing this identity in (2), we have that:

$$C_1^4 - 2MC_1^2 - 8NC_1 + \left(M^2 - 4NL\right) = 0$$

This equation is similar to (4), such that:

$$X = C_1$$
$$P = -2M$$
$$Q = -8N$$
$$R = M^2 - 4NL$$

So: $M = -\frac{P}{2}$ $N = -\frac{Q}{8}$ $L = \left(R - \frac{P^2}{4}\right)\frac{4}{Q}$
These are the coefficients of the cubic auxiliary solvent equation:

$$Y_1^3 - LY_1^2 + MY_1 - N = 0$$

It‘s roots are a^2, b^2 y c^2.
So from the equation (4): $X = C_1 = \sqrt{a^2b^2} + \sqrt{a^2c^2} + \sqrt{b^2c^2}$

1. **THIRD SOLUTION TO THE FOURTH GRADE EQUATION**

Consider the fourth degree equation: $X^4 + C_0X^3 + C_1X^2 + C_2X + C_3 = 0$
Now divide the equation in X^2:

$$\frac{X^4 + C_0X^3 + C_1X^2 + C_2X + C_3}{X^2} = X^2 + C_0X + C_1 + \frac{C_2}{X} + \frac{C_3}{X^2} = 0$$

Treat this last equation as an almost-symmetric equation, and grouping, this is so:

$$\left(X^2 + \frac{C_3}{X^2}\right) + \left(C_0X + \frac{C_2}{X}\right) + C_1 = 0 \tag{5}$$

Now: $\left(C_0X + \frac{C_2}{X}\right)^2 = C_0^2X^2 + 2C_0C_2 + \frac{C_2^2}{X^2} = \left(C_0^2X^2 + \frac{C_2^2}{X^2}\right) + 2C_0C_2$
Suppose that the coefficients comply with the proportion: $\frac{C_2^2}{C_0^2} = C_3$
So that: $\left(C_0^2X^2 + \frac{C_2^2}{X^2}\right) + 2C_0C_2 = C_0^2\left(X^2 + \frac{C_3}{X^2}\right) + 2C_0C_2$
And: $C_0X + \frac{C_2}{X} = Y$
Therefore: $Y^2 = C_0^2\left(X^2 + \frac{C_3}{X^2}\right) + 2C_0C_2$
So: $X^2 + \frac{C_3}{X^2} = \frac{Y^2 - 2C_0C_2}{C_0^2}$
Replacing this in equation (5):

$$\frac{Y^2 - 2C_0C_2}{C_0^2} + Y + C_1 = Y^2 + C_0^2Y + (C_0^2C_1 - 2C_0C_2) = 0$$

this is a second degree auxiliary equation, and it‘s roots are:

$$Y_1 = -\frac{C_0^2}{2} + \sqrt{2C_0C_2 - C_0^2C_1 + \frac{C_0^4}{4}}$$

$$Y_2 = -\frac{C_0^2}{2} - \sqrt{2C_0C_2 - C_0^2C_1 + \frac{C_0^4}{4}}$$

These roots allow to find the values of $Y = C_0X + \frac{C_2}{X}$
Solving the second degree auxiliary equations:

$$X^2 - \frac{Y_1}{C_0}X + \frac{C_2}{C_0} = 0$$

$$X^2 - \frac{Y_2}{C_0}X + \frac{C_2}{C_0} = 0$$

We have the four solutions to the fourth degree equation $X^4 + C_0X^3 + C_1X^2 + C_2X + C_3 = 0$:

$$X_1 = -\frac{Y_1}{2C_0} + \sqrt{\frac{Y_1^2}{4C_0^2} - \frac{C_2}{C_0}} X_2 = -\frac{Y_1}{2C_0} - \sqrt{\frac{Y_1^2}{4C_0^2} - \frac{C_2}{C_0}} X_3 = -\frac{Y_2}{2C_0} + \sqrt{\frac{Y_2^2}{4C_0^2} - \frac{C_2}{C_0}} \quad X_4 = -\frac{Y_2}{2C_0} - \sqrt{\frac{Y_2^2}{4C_0^2} - \frac{C_2}{C_0}}$$

What remains to be done is to ensure that every equation of the fourth degree has the proportion in its coefficients $\frac{C_2^2}{C_0^2} = C_3$, for this, it must undergo a transformation using Taylor‘s formula, which in the case of the equation of fourth degree is:

$$f(x+h) = (x+h)^4 + \frac{f^{(3)}(h)(x+h)^3}{(3)!} + \frac{f^{(2)}(h)(x+h)^2}{(2)!} + f^{(1)}(h)(x+h) + f(h)$$

Then, to fulfill the proportion: $\frac{\left(f^{(1)}(h)\right)^2}{\left(\frac{f^{(3)}(h)}{(3)!}\right)^2} = f(h) \rightarrow \left(f^{(1)}(h)\right)^2 = \left(\frac{f^{(3)}(h)}{(3)!}\right)^2 f(h)$

And:

$$f^{(1)}(h) = 4h^3 + 3C_0h^2 + 2C_1h + C_2$$

$$\frac{f^{(3)}(h)}{(3)!} = \frac{24h + 6C_0}{6} = 4h + C_0$$

$$f(h) = h^4 + C_0h^3 + C_1h^2 + C_2h + C_3$$

So, solving and eliminating similar terms we have that $\left(f^{(1)}(h)\right)^2 = \left(\frac{f^{(3)}(h)}{(3)!}\right)^2 f(h)$ is:

$$h^3 + \left(\frac{16C_3 + 2C_0C_2 + C_1C_0^2 - 4C_1^2}{8C_2 - 4C_0C_1 + C_0^3}\right)h^2 + \left(\frac{8C_0C_3 + C_0^2C_2 - 2C_1C_2}{8C_2 - 4C_0C_1 + C_0^3}\right)h + \left(\frac{C_0^2C_3 - C_2^2}{8C_2 - 4C_0C_1 + C_0^3}\right) = 0$$

It is a third-degree equation that gives three roots h, to transform any fourth-degree equation into one that has the proportion $\frac{C_2^2}{C_0^2} = C_3$.

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