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# Some Fractional Integral Inequalities for Stochastic Processes whose First and Second Derivatives are Quasi-Convex

# Algunas Desigualdades Integrales Fraccionarias para Procesos Estocásticos cuyas Primeras y Segundas Derivadas son Quasi-Convexas

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**Abstract**

This work contains some Riemman-Liouville fractional integral inequalities of Hermite-Hadamard and Ostrowski type involving first and second derivatives of stochastic processes that are quasi convex. Also, from the attained results are deduced similar inequalities corresponding to the integral of Riemman.

*Keywords:* Fractional Integral Inequalities; Stochastic Processes; Quasi Convexity.
*2008 MSC:* 35A23, 60E15, 26A33

**Resumen**

Este artÃculo contiene algunas desigualdades integrales fraccionarias de Riemann-Liouville del tipo Hermite-Hadamard y Ostrowski que involucran primeras y segundas derivadas de procesos estocásticos que son quasi-convexas. TambiÃ©n, de los resultados obtenidos se deducen desigualdades similares para la integral de Riemann.

*Palabras claves:* Desigualdades integrales fraccionarias; Procesos Estocásticos; Quasi-convexidad.
*2008 MSC:* 35A23, 60E15, 26A33

## 1. Introduction

In 1974, B. Nagy ([19]) applied a characterization of measurable stochastic processes to solving a generalization of the (additive) Cauchy functional equation, thus beginning the study on convex stochastic processes. Later, in 1980 Nikodem [21] considered convex stochastic processes, in 1995 Skowronski [30] obtained some further results on convex stochastic processes, which generalize some known properties of convex functions and M. Shaked and J. Shantikumar in [27] wrote about its applications. For other results related to stochastic processes see [3],[5],[16],[27], [28] where further references are given.

Also the evolution of the concept of convexity has had a great impact in the community of investigators. In recent years, for example, generalized concepts such as log-convexity, $s$-convexity , $P$-convexity, $\eta$-convexity, quasi convexity, $MT$-convexity , $h$-convexity and others , as well as combinations of these new concepts have been introduced. In the following references there is more information about this topic [1], [2], [4], [8], [11], [15],[32].

A. Guerraggio and E. Molho in [12] wrote that the concept of quasi convexity is attributed to Bruno de Finetti for its publication of the year 1949 [7] where it uses these functions and with W. Fenchel who in [6] employs for the first time the denomination of quasi convex. However, a formalization appeared 20 years before in a famous work on theory of games, where one raises the so-called Theorem of Minimax, published in 1928 by the hungarian John von Neumann [20]. In recent years H. Greenberg and W. Pierskalla in [11], and E. Set et.al. in [26] have written about quasi-convexity and its applications to Simpsom type inequalities.

The well-known Hermite-Hadamard inequality reads that for every convex function $f : I \subset \mathbb{R} \to \mathbb{R}$ it is had that

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

holds for every $a, b \in I$ with $a < b$ (See [18]); and A. Ostrowski established that for every function $f : I \subset [0, \infty) \to \mathbb{R}$, differentiable on $int(I)$ such that $f' \in L([a, b])$, where $a, b \in I$ with $a < b$, if $|f'(x)| \leq M$ then the following inequality holds

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{M}{b-a}\left(\frac{(x-a)^2 + (b-x)^2}{2}\right), \tag{2}$$

this last is known as the Ostrowski Inequality (See [22]).

Many researchers have developed works where they relate the concepts of generalized convexity and stochastic processes using the inequalities (1) and (2), for example, E. Set et. al. in [25] investigated Hermite-Hadamard type inequalities for stochastic processes in the second sense, and Vivas-Cortez in [33] studied about $(h_1, h_2, m) - GA$-convexity for stochastic Processes.

Motivated by the works of the authors cited, the objective of this research is to present some results regarding fractional integrals inequalities, specifically some of the type Hermite-Hadamard and Ostrowski, involving first and second derivatives of stochastic processes that are quasi convex.

## 2. Preliminaries

In this section it is presented some basics of calculus for stochastic processes and the Riemann-Liouville fractional integral as a theoretical framework for the development of this work.

### *2.1. About Calculus of Stochastic Processes*

**Definition 2.1.** *Let $(\Omega, \mathcal{A}, P)$ be an arbitrary probability space. A function $X : \Omega \to \mathbb{R}$ is called a random variable iff it is $\mathcal{A}$-measurable and $P\{w \in \Omega : X(w) \notin \mathbb{R}\} = 0$. Let $I \subset \mathbb{R}$ be time. A function $X : I \times \Omega \to \mathbb{R}$ is called a stochastic process iff for all $t \in T$ the function $X(t, \cdot) : \Omega \to \mathbb{R}$ is a random variable.*

In this work $I$ is an interval and $X(t, \cdot)$ is called a stochastic process with continuous time.

All the following definitions can be found in the following references [13], [16],[30].

**Definition 2.2.** *Let $(\Omega, A, P)$ be a probability space. It is said that the stochastic process $X : I \times \Omega \to \mathbb{R}$ is called*

1. *Continuous in probability on the interval $I$ if for all $t_0 \in I$ it is had that*

$$P - \lim_{t \to t_0} X(t, \cdot) = X(t_0, \cdot),$$

*where $P - lim$ denotes the limit in probability;*

2. *Mean-square continuous in the interval $I$ if for all $t_0 \in I$*

$$P - \lim_{t \to t_0} \mathbb{E}(X(t, \cdot) - X(t_0, \cdot)) = 0,$$

*where $\mathbb{E}(X(t, \cdot))$ denote the expectation value of the random variable $X(t, \cdot)$;*

3. *Increasing (decreasing) if for all $u, v \in I$ such that $t < s$,*

$$X(u, \cdot) \leq X(v, \cdot), \quad (X(u, \cdot) \geq X(v, \cdot)) \qquad (a.e)$$

4. *Monotonic if it is increasing or decreasing;*
5. *Differentiable at a point $t \in I$ if there is a random variable*

$$X'(t, \cdot) : I \times \Omega \to \mathbb{R}, \text{ such that} X'(t, \cdot) = P - \lim_{t \to t_0} \frac{X(t, \cdot) - X(t_0, \cdot)}{t - t_0}.$$

We say that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is continuous (differentiable) is it is continuous (differentiable) at every point of the interval $I$.

In the book of K. Sobczyk [31] it is found the following.

**Definition 2.3.** *Let $(\Omega, A, P)$ be a probability space $I \subset \mathbb{R}$ be an interval with $E(X(t)^2) < \infty$ for all $t \in I$. Let $[a, b] \subset I, a = t_0 < t_1 < ... < t_n = b$ be a partition of $[a, b]$ and $\theta_k \in [t_{k-1}, t_k]$ for $k = 1, 2, ..., n$. A random variable $Y : \Omega \to \mathbb{R}$ is called mean-square integral of the process $X(t, \cdot)$ on $[a, b]$ if the following identity holds:*

$$\lim_{n \to \infty} E \left[ \sum_{k=0}^{n} X(\theta_k, \cdot)(t_k - t_{k-1}) - Y \right]^2 = 0$$

*Then it can be written*

$$\int_a^b X(t, \cdot) dt = Y(\cdot) \qquad (a.e.).$$

Also, mean square integral operator is increasing, that is,

$$\int_a^b X(t, \cdot) dt \leq \int_a^b Z(t, \cdot) dt (a.e.)$$

where $X(t, \cdot) \leq Z(t, \cdot)$ in $[a, b]$ ([29]).
In throughout paper, we will consider the stochastic processes that is with continuous time and mean-square continuous.

**Definition 2.4.** *([21]) Set $(\Omega, \mathcal{A}, P)$ be a probability space and $I \subset \mathbb{R}$ be an interval. It is said that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is convex if the following inequality holds almost everywhere*

$$X(\lambda u + (1 - \lambda) v, \cdot) \leq \lambda X(u, \cdot) + (1 - \lambda) X(v, \cdot) \tag{3}$$

*for all $u, v \in I$ and $\lambda \in [0, 1]$.*

Now, we give the well-known Hermite-Hadamard integral inequality for convex stochastic processes (see [13]).

**Theorem 2.5.** *If $X : I \times \Omega \to \mathbb{R}$ is Jensen-convex and mean square continuous in the interval $T \times \Omega$, then for any $u, v \in T$, we have*

$$X\left(\frac{u+v}{2}, \cdot\right) \leq \frac{1}{u-v}\int_u^v X(t,\cdot)dt \leq \frac{X(u,\cdot)+X(v,\cdot)}{2}$$

The following definitions will be the base for our results (See [14]).

**Definition 2.6.** *It is said that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is a quasi convex stochastic process if the following inequality holds almost everywhere*

$$X(ta+(1-t)b,\cdot) \leq \text{máx}\{X(a,\cdot), X(b,\cdot)\}$$

*for all $a, b \in I$ and $t \in [0,1]$.*

*2.2. About Riemann-Liouville Fractional Integral.*

Before we establish our main results, we give some necessary de nitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. For more details, one can consult [10, 17, 23].

**Definition 2.7.** *Let$f \in L_1([a,b])$. The Riemann-Liouville integrals $J^\alpha_{a+}$ and $J^\alpha_{b-}$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt$$

*and*

$$J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt$$

*respectively, where $\Gamma(\alpha)$ is the Gamma funcion and $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.*

Using the Riemann-Liouville fractional integral, Sarikaya et all [24], stablished the Hermite-Hadamard inequalities version.

**Theorem 2.8.** *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $a < b$ and $f \in L_1([a,b])$. If $f$ is a convex function on $[a,b]$ then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left(J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)\right) \leq \frac{f(a)+f(b)}{2}$$

*with $\alpha > 0$.*

It will be necessary the following Lemma established by J. Gomez and J. Hernández in [9] for the development of this work.

**Lemma 2.9.** *Let $X : I \times \Omega \to \mathbb{R}$ be a twice differentiable stochastic process on int$(I)$, where $I \subset \mathbb{R}$ is an interval, $a, b \in$ int$(I)$ with $a < b$. If $X''$ is mean square integrable on $[a,b]$ then the following equality holds:*

$$\frac{2^{\alpha-1}\Gamma(\alpha+2)}{(b-a)^\alpha}\left(J^\alpha_{\frac{a+b}{2}+}X(b,\cdot) + J^\alpha_{\frac{a+b}{2}-}X(a,\cdot)\right) - (\alpha+1)X\left(\frac{a+b}{2},\cdot\right)$$
$$\leq \frac{(b-a)^2}{8}\left[\int_0^1 t^{\alpha+1}X''\left(\frac{t}{2}a + \frac{2-t}{2}b,\cdot\right)dt + \int_0^1 t^{\alpha+1}X''\left(\frac{2-t}{2}a + \frac{t}{2}b,\cdot\right)dt\right]$$

## 3. Main Results

### *3.1. About Hermite-Hadamard type inequalities*

**Theorem 3.1.** *Let $X : I \times \Omega \to \mathbb{R}$ be a twice differentiable stochastic process on int($I$), where $I \subset \mathbb{R}$ is an interval, $a, b \in$ int($I$) with $a < b$. If $X''$ is mean square integrable on $[a, b]$ and quasi convex on $[a, b]$ then the following inequality holds almost everywhere*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+2)}{(b-a)^{\alpha}}\left(J^{\alpha}_{\frac{a+b}{2}+}X(b,\cdot)+J^{\alpha}_{\frac{a+b}{2}-}X(a,\cdot)\right)-(\alpha+1)\,X\left(\frac{a+b}{2},\cdot\right)\right|$$
$$\leq \frac{(b-a)^2}{4(\alpha+2)}\,\text{máx}\left\{\left|X''(a,\cdot)\right|,\left|X''(b,\cdot)\right|\right\}.$$

**Proof.** Using Lemma 2.9 and the triangular inequality it is obtained

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+2)}{(b-a)^{\alpha}}\left(J^{\alpha}_{\frac{a+b}{2}+}X(b,\cdot)+J^{\alpha}_{\frac{a+b}{2}-}X(a,\cdot)\right)-(\alpha+1)\,X\left(\frac{a+b}{2},\cdot\right)\right|$$
$$\leq \frac{(b-a)^2}{8}\left[\left|\int_0^1 t^{\alpha+1}X''\left(\frac{t}{2}a+\frac{2-t}{2}b,\cdot\right)dt\right|+\left|\int_0^1 t^{\alpha+1}X''\left(\frac{2-t}{2}a+\frac{t}{2}b,\cdot\right)dt\right|\right].$$

Since $|X''|$ is a quasi convex stochastic process it is had that

$$\left|X''\left(\frac{t}{2}a+\frac{2-t}{2}b,\cdot\right)\right|\leq \text{máx}\left\{\left|X''(a,\cdot)\right|,\left|X''(b,\cdot)\right|\right\}$$

and

$$\left|X''\left(\frac{2-t}{2}a+\frac{t}{2}b,\cdot\right)\right|\leq \text{máx}\left\{\left|X''(a,\cdot)\right|,\left|X''(b,\cdot)\right|\right\}.$$

Therefore,

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+2)}{(b-a)^{\alpha}}\left(J^{\alpha}_{\frac{a+b}{2}+}X(b,\cdot)+J^{\alpha}_{\frac{a+b}{2}-}X(a,\cdot)\right)-(\alpha+1)\,X\left(\frac{a+b}{2},\cdot\right)\right|$$

$$\begin{aligned}&\leq \frac{2\,(b-a)^2}{8}\int_0^1 t^{\alpha+1}\,\text{máx}\left\{\left|X''(a,\cdot)\right|,\left|X''(b,\cdot)\right|\right\}dt\\ &= \frac{(b-a)^2}{4(\alpha+2)}\,\text{máx}\left\{\left|X''(a,\cdot)\right|,\left|X''(b,\cdot)\right|\right\}.\end{aligned}$$

The proof is complete. ■

**Corollary 3.2.** *Let $X : I \times \Omega \to \mathbb{R}$ be a twice differentiable stochastic process on int($I$), where $I \subset \mathbb{R}$ is an interval, $a, b \in$ int($I$) with $a < b$. If $X''$ is mean square integrable on $[a, b]$ and quasi convex on $[a, b]$ then the following inequality holds almost everywhere*

$$\left|\frac{1}{(b-a)}\int_a^b X(t,\cdot)dt-X\left(\frac{a+b}{2},\cdot\right)\right|\leq\frac{(b-a)^2}{24}\,\text{máx}\left\{\left|X''(a,\cdot)\right|,\left|X''(b,\cdot)\right|\right\}.$$

**Theorem 3.3.** *Let $X : I \times \Omega \to \mathbb{R}$ be a twice differentiable stochastic process on int$(I)$, where $I \subset \mathbb{R}$ is an interval, $a, b \in$ int$(I)$ with $a < b$, If $|X''|^q$ is mean square integrable on $[a, b]$ and quasi convex on $[a, b]$ for $q > 1$, then the following inequality holds almost everywhere*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+2)}{(b-a)^{\alpha}}\left(J^{\alpha}_{\frac{a+b}{2}+}X(b,\cdot)+J^{\alpha}_{\frac{a+b}{2}-}X(a,\cdot)\right)-(\alpha+1)X\left(\frac{a+b}{2},\cdot\right)\right|$$
$$\leq \frac{(b-a)^2}{4}\left(\frac{(\alpha+1)q}{q-1}\right)^{1/q-1}\text{máx}\left\{\left|X''(a,\cdot)\right|^q,\left|X''(b,\cdot)\right|^q\right\}$$

**Proof.** From Lemma 2.9 and using the Hölder inequality we have

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+2)}{(b-a)^{\alpha}}\left(J^{\alpha}_{\frac{a+b}{2}+}X(b,\cdot)+J^{\alpha}_{\frac{a+b}{2}-}X(a,\cdot)\right)-(\alpha+1)X\left(\frac{a+b}{2},\cdot\right)\right|$$
$$\leq \frac{(b-a)^2}{8}\left(\int_0^1 t^{(\alpha+1)p}\right)^{1/p}\left[\left(\int_0^1\left|X''\left(\frac{t}{2}a+\frac{2-t}{2}b,\cdot\right)\right|^q dt\right)^{1/q}+\left(\int_0^1\left|X''\left(\frac{2-t}{2}a+\frac{t}{2}b,\cdot\right)\right|^q dt\right)^{1/q}\right]$$

where $1/p = 1 - 1/q$.

Since $|X|^q$ is quasi convex on $[a, b]$ then

$$\left|X''\left(\frac{t}{2}a+\frac{2-t}{2}b,\cdot\right)\right|^q \leq \text{máx}\left\{\left|X''(a,\cdot)\right|^q,\left|X''(b,\cdot)\right|^q\right\}$$

and

$$\left|X''\left(\frac{2-t}{2}a+\frac{t}{2}b,\cdot\right)\right|^q \leq \text{máx}\left\{\left|X''(a,\cdot)\right|^q,\left|X''(b,\cdot)\right|^q\right\}.$$

It is deduced

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+2)}{(b-a)^{\alpha}}\left(J^{\alpha}_{\frac{a+b}{2}+}X(b,\cdot)+J^{\alpha}_{\frac{a+b}{2}-}X(a,\cdot)\right)-(\alpha+1)X\left(\frac{a+b}{2},\cdot\right)\right|$$
$$\leq \frac{2(b-a)^2}{8}\left(\int_0^1 t^{(\alpha+1)p}\right)^{1/p}\left(\int_0^1\text{máx}\left\{\left|X''(a,\cdot)\right|^q,\left|X''(b,\cdot)\right|^q\right\}dt\right)^{1/q}$$
$$\leq \frac{(b-a)^2}{4((\alpha+1)p)^{1/p}}\text{máx}\left\{\left|X''(a,\cdot)\right|,\left|X''(b,\cdot)\right|\right\}$$

The proof is complete. ■

**Corollary 3.4.** *Let $X : I \times \Omega \to \mathbb{R}$ be a twice differentiable stochastic process on int$(I)$, where $I \subset \mathbb{R}$ is an interval, $a, b \in$ int$(I)$ with $a < b$, If $|X''|^q$ is mean square integrable on $[a, b]$ and quasi convex on $[a, b]$ for $q > 1$, then the following inequality holds almost everywhere*

$$\left|\frac{1}{(b-a)}\int_a^b X(t,\cdot)dt - X\left(\frac{a+b}{2},\cdot\right)\right| \leq \frac{(b-a)^2}{4}\left(\frac{q}{q-1}\right)^{1/q-1}\text{máx}\left\{\left|X''(a,\cdot)\right|^q,\left|X''(b,\cdot)\right|^q\right\}$$

**Theorem 3.5.** *Let $X : I \times \Omega \to \mathbb{R}$ be a twice differentiable stochastic process on int$(I)$, where $I \subset \mathbb{R}$ is an interval, $a, b \in$ int$(I)$ with $a < b$, If $|X''|^q$ is mean square integrable on $[a,b]$ and quasi convex on $[a,b]$ for $q > 1$, then the following inequality holds almost everywhere*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+2)}{(b-a)^\alpha}\left(J^\alpha_{\frac{a+b}{2}+}X(b,\cdot)+J^\alpha_{\frac{a+b}{2}-}X(a,\cdot)\right)-(\alpha+1)X\left(\frac{a+b}{2},\cdot\right)\right|$$
$$\leq \frac{(b-a)^2}{4(\alpha+1)}\text{máx}\left\{\left|X''(a,\cdot)\right|,\left|X''(b,\cdot)\right|\right\}$$

**Proof.** From Lemma 2.9 and using the power mean inequality for $q \geq 1$ and the quasi convexity of $|X|^q$ it is had that

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+2)}{(b-a)^\alpha}\left(J^\alpha_{\frac{a+b}{2}+}X(b,\cdot)+J^\alpha_{\frac{a+b}{2}-}X(a,\cdot)\right)-(\alpha+1)X\left(\frac{a+b}{2},\cdot\right)\right|\\
&\leq \frac{(b-a)^2}{8}\left[\left|\int_0^1 t^{\alpha+1}X''\left(\frac{t}{2}a+\frac{2-t}{2}b,\cdot\right)dt\right|+\left|\int_0^1 t^{\alpha+1}X''\left(\frac{2-t}{2}a+\frac{t}{2}b,\cdot\right)dt\right|\right]\\
&\leq \frac{(b-a)^2}{8}\left(\int_0^1 t^{\alpha+1}dt\right)^{1-1/q}\left[\left(\int_0^1 t^{\alpha+1}\left|X''\left(\frac{t}{2}a+\frac{2-t}{2}b,\cdot\right)\right|^q dt\right)^{1/q}\right.\\
&\qquad\qquad\left.+\left(\int_0^1 t^{\alpha+1}\left|X''\left(\frac{2-t}{2}a+\frac{t}{2}b,\cdot\right)\right|^q dt\right)^{1/q}\right]\\
&\leq \frac{(b-a)^2}{4}\left(\int_0^1 t^{\alpha+1}dt\right)\left(\text{máx}\left\{\left|X''(a,\cdot)\right|^q,\left|X''(b,\cdot)\right|^q\right\}\right)^{1/q}\\
&= \frac{(b-a)^2}{4(\alpha+1)}\left(\text{máx}\left\{\left|X''(a,\cdot)\right|^q,\left|X''(b,\cdot)\right|^q\right\}\right)^{1/q}\\
&= \frac{(b-a)^2}{4(\alpha+1)}\text{máx}\left\{\left|X''(a,\cdot)\right|,\left|X''(b,\cdot)\right|\right\}
\end{aligned}$$

The proof is complete. ■

**Corollary 3.6.** *Let $X : I \times \Omega \to \mathbb{R}$ be a twice differentiable stochastic process on int$(I)$, where $I \subset \mathbb{R}$ is an interval, $a, b \in$ int$(I)$ with $a < b$, If $|X''|^q$ is mean square integrable on $[a,b]$ and quasi convex on $[a,b]$ for $q > 1$, then the following inequality holds almost everywhere*

$$\left|\frac{1}{(b-a)}\int_a^b X(t,\cdot)dt - X\left(\frac{a+b}{2},\cdot\right)\right| \leq \frac{(b-a)^2}{8}\text{máx}\left\{\left|X''(a,\cdot)\right|,\left|X''(b,\cdot)\right|\right\}$$

*3.2. About Ostrowski inequalties type*

**Lemma 3.7.** *Let $X : [a,b] \times \Omega \to \mathbb{R}$ be a differentiable stochastic process on $(a,b)$ with $a < b$ . If $X'$ is mean square integrable on $[a,b]$, then for all $x \in [a,b]$ and $\alpha > 0$ the following equality holds*

$$\left(\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}\right)X(x,\cdot)-\frac{\Gamma(\alpha+1)}{b-a}\left(\left(J^\alpha_{x-}X\right)(a,\cdot)+\left(J^\alpha_{x+}X\right)(b,\cdot)\right)$$
$$= \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^\alpha X'(tx+(1-t)a,\cdot)\,dt - \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^\alpha X'(tx+(1-t)b,\cdot)\,dt$$

**Proof.** Integrating by parts

$$
\begin{aligned}
&\int_0^1 t^\alpha X'\left(tx+(1-t)a,\cdot\right)dt\\
&\qquad= \left.\frac{t^\alpha X\left(tx+(1-t)a,\cdot\right)}{x-a}\right|_0^1 - \frac{\alpha}{x-a}\int_0^1 t^{\alpha-1}X\left(tx+(1-t)a,\cdot\right)dt\\
&\qquad= \frac{X\left(x,\cdot\right)}{x-a} - \frac{\alpha}{(x-a)^\alpha}\int_a^x (u-a)^{\alpha-1}X\left(u,\cdot\right)du
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 t^\alpha X'\left(tx+(1-t)b,\cdot\right)dt\\
&\qquad= \left.\frac{-t^\alpha X\left(tx+(1-t)b,\cdot\right)}{b-x}\right|_0^1 + \frac{\alpha}{b-x}\int_0^1 t^{\alpha-1}X\left(tx+(1-t)b,\cdot\right)dt\\
&\qquad= \frac{-X\left(x,\cdot\right)}{b-x} + \frac{\alpha}{(b-x)^\alpha}\int_x^b (b-u)^{\alpha-1}X\left(u,\cdot\right)du
\end{aligned}
$$

Multiplying the first integral by $(x-a)^{\alpha+1}/(b-a)$ and the second integral by $(b-x)^{\alpha+1}/(b-a)$, and after subtracting it is had that

$$
\begin{aligned}
&\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^\alpha X'\left(tx+(1-t)a,\cdot\right)dt - \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^\alpha X'\left(tx+(1-t)b,\cdot\right)dt\\
&= \left(\frac{(x-a)^\alpha X\left(x,\cdot\right)}{b-a} - \frac{(x-a)^\alpha\alpha}{b-a}\int_a^x (u-a)^{\alpha-1}X\left(u,\cdot\right)du\right)\\
&\qquad\qquad -\left(\frac{-(b-x)^\alpha X\left(x,\cdot\right)}{b-a} - \frac{(b-x)\alpha}{b-a}\int_x^b (u-b)^{\alpha-1}X\left(u,\cdot\right)du\right)\\
&= \left(\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}\right)X\left(x,\cdot\right) - \frac{(x-a)^\alpha\alpha}{b-a}\int_a^x (u-a)^{\alpha-1}X\left(u,\cdot\right)du\\
&\qquad\qquad + \frac{(b-x)\alpha}{b-a}\int_x^b (u-b)^{\alpha-1}X\left(u,\cdot\right)du\\
&= \left(\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}\right)X\left(x,\cdot\right) - \frac{(x-a)^\alpha\Gamma(\alpha+1)}{b-a}\left(J_{x-}^\alpha X\right)\left(a,\cdot\right)\\
&\qquad\qquad + \frac{(b-x)^\alpha\Gamma(\alpha+1)}{b-a}\left(J_{x+}^\alpha X\right)\left(b,\cdot\right).
\end{aligned}
$$

The proof is complete. ■

**Theorem 3.8.** *Let $X : [a,b] \times \Omega \to \mathbb{R}$ be a differentiable stochastic process on $(a,b)$ with $a < b$ , and $X'$ is a mean square integrable on $[a,b]$ . If $|X'|$ is a quasi convex stochastic process and $|X'(x,\cdot)| \leq M$ then, for $\alpha > 0$ the following inequality holds almost everywhere*

$$\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)X(x,\cdot)-\frac{\Gamma(\alpha+1)}{b-a}\left(\left(J^{\alpha}_{x-}X\right)(a,\cdot)+\left(J^{\alpha}_{x+}X\right)(b,\cdot)\right)\right|$$
$$\leq \frac{1}{(b-a)(\alpha+1)}\left[(x-a)^{\alpha+1}\operatorname{máx}\left\{M,\left|X'(a,\cdot)\right|\right\}+(b-x)^{\alpha+1}\operatorname{máx}\left\{M,\left|X'(b,\cdot)\right|\right\}\right].$$

**Proof.** Using Lemma 3.7 and the triangular inequality it is had that

$$\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)X(x,\cdot)-\frac{\Gamma(\alpha+1)}{b-a}\left(\left(J^{\alpha}_{x-}X\right)(a,\cdot)+\left(J^{\alpha}_{x+}X\right)(b,\cdot)\right)\right|$$
$$\leq \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left|X'\left(tx+(1-t)a,\cdot\right)\right|dt+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left|X'\left(tx+(1-t)b,\cdot\right)\right|dt$$

Since $|X'|$ is a quasi convex stochastic process and the fact that $|X'(x,\cdot)| \leq M$ it is obtained

$$\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)X(x,\cdot)-\frac{\Gamma(\alpha+1)}{b-a}\left(\left(J^{\alpha}_{x-}X\right)(a,\cdot)+\left(J^{\alpha}_{x+}X\right)(b,\cdot)\right)\right|$$
$$\leq \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\operatorname{máx}\left\{\left|X'(x,\cdot)\right|,\left|X'(a,\cdot)\right|\right\}dt+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\operatorname{máx}\left\{\left|X'(x,\cdot)\right|,\left|X'(b,\cdot)\right|\right\}dt$$
$$= \frac{(x-a)^{\alpha+1}}{(b-a)(\alpha+1)}\operatorname{máx}\left\{\left|X'(x,\cdot)\right|,\left|X'(a,\cdot)\right|\right\}+\frac{(b-x)^{\alpha+1}}{(b-a)(\alpha+1)}\operatorname{máx}\left\{\left|X'(x,\cdot)\right|,\left|X'(b,\cdot)\right|\right\}$$
$$\leq \frac{(x-a)^{\alpha+1}}{(b-a)(\alpha+1)}\operatorname{máx}\left\{M,\left|X'(a,\cdot)\right|\right\}+\frac{(b-x)^{\alpha+1}}{(b-a)(\alpha+1)}\operatorname{máx}\left\{M,\left|X'(b,\cdot)\right|\right\}$$

The proof is complete. ■

**Corollary 3.9.** *Let $X : [a,b] \times \Omega \to \mathbb{R}$ be a differentiable stochastic process on $(a,b)$ with $a < b$ , and $X'$ is a mean square integrable on $[a,b]$ . If $|X'|$ is a quasi convex stochastic process and $|X'(x,\cdot)| \leq M$ then, for $\alpha > 0$ the following inequality holds almost everywhere*

$$\left|X(x,\cdot)-\frac{1}{b-a}\int_a^b X(t,\cdot)dt\right|$$
$$\leq \frac{1}{2(b-a)}\left[(x-a)^2\operatorname{máx}\left\{M,\left|X'(a,\cdot)\right|\right\}+(b-x)^2\operatorname{máx}\left\{M,\left|X'(a,\cdot)\right|\right\}\right].$$

**Theorem 3.10.** *Let $X : [a,b] \times \Omega \to \mathbb{R}$ be a differentiable stochastic process on $(a,b)$ with $a < b$ , $p > 1$ and $X'$ is a mean square integrable on $[a,b]$ . If $|X'|$ is a quasi convex stochastic process and $|X'(x,\cdot)| \leq M$ then, for $\alpha > 0$ the following inequality holds almost everywhere*

$$\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)X(x,\cdot)-\frac{\Gamma(\alpha+1)}{b-a}\left(\left(J^{\alpha}_{x-}X\right)(a,\cdot)+\left(J^{\alpha}_{x+}X\right)(b,\cdot)\right)\right|$$
$$\leq \frac{(x-a)^{\alpha+1}}{(b-a)(\alpha+1)^{1/p}}\operatorname{máx}\left\{M,\left|X'(a,\cdot)\right|\right\}+\frac{(b-x)^{\alpha+1}}{(b-a)(\alpha+1)^{1/p}}\operatorname{máx}\left\{M,\left|X'(b,\cdot)\right|\right\}$$

**Proof.** Using Lemma 3.7, the triangular inequality and Hölder inequality it is had that

$$\begin{aligned}
&\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)X(x,\cdot)-\frac{\Gamma(\alpha+1)}{b-a}\left((J_{x-}^{\alpha}X)(a,\cdot)+(J_{x+}^{\alpha}X)(b,\cdot)\right)\right|\\
&\leq \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left|X'(tx+(1-t)a,\cdot)\right|dt+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left|X'(tx+(1-t)b,\cdot)\right|dt\\
&\leq \frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 t^{\alpha p}dt\right)^{1/p}\left(\int_0^1\left|X'(tx+(1-t)a,\cdot)\right|^q dt\right)^{1/q}\\
&\quad+\frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 t^{\alpha p}dt\right)^{1/p}\left(\int_0^1\left|X'(tx+(1-t)b,\cdot)\right|^q dt\right)^{1/q}
\end{aligned}$$

where $1/q = 1 - 1/p$.

Since $|X'|$ is a quasi convex stochastic process and the fact that $|X'(x,\cdot)| \leq M$ it is had that

$$\begin{aligned}
&\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)X(x,\cdot)-\frac{\Gamma(\alpha+1)}{b-a}\left((J_{x-}^{\alpha}X)(a,\cdot)+(J_{x+}^{\alpha}X)(b,\cdot)\right)\right|\\
&\leq \frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 t^{\alpha p}dt\right)^{1/p}\left(\int_0^1\left[\text{máx}\left\{\left|X'(x,\cdot)\right|,\left|X'(a,\cdot)\right|\right\}\right]^q dt\right)^{1/q}\\
&\quad+\frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 t^{\alpha p}dt\right)^{1/p}\left(\int_0^1\left[\text{máx}\left\{\left|X'(x,\cdot)\right|,\left|X'(b,\cdot)\right|\right\}\right]^q dt\right)^{1/q}\\
&=\frac{(x-a)^{\alpha+1}}{(b-a)(\alpha+1)^{1/p}}\,\text{máx}\left\{\left|X'(x,\cdot)\right|,\left|X'(a,\cdot)\right|\right\}+\frac{(b-x)^{\alpha+1}}{(b-a)(\alpha+1)^{1/p}}\,\text{máx}\left\{\left|X'(x,\cdot)\right|,\left|X'(b,\cdot)\right|\right\}\\
&=\frac{(x-a)^{\alpha+1}}{(b-a)(\alpha+1)^{1/p}}.\,\text{máx}\left\{M,\left|X'(a,\cdot)\right|\right\}+\frac{(b-x)^{\alpha+1}}{(b-a)(\alpha+1)^{1/p}}\,\text{máx}\left\{M,\left|X'(b,\cdot)\right|\right\}
\end{aligned}$$

The proof is complete. ■

**Corollary 3.11.** *Let $X : [a,b]\times\Omega \to \mathbb{R}$ be a differentiable stochastic process on $(a,b)$ with $a < b$ , and $X'$ is a mean square integrable on $[a,b]$ . If $|X'|$ is a quasi convex stochastic process and $|X'(x,\cdot)| \leq M$ then, for $\alpha > 0$ the following inequality holds almost everywhere*

$$\begin{aligned}
&\left|X(x,\cdot)-\frac{1}{b-a}\int_a^b X(t,\cdot)dt\right|\\
&\leq \frac{(x-a)}{2^{1/p}(b-a)}.\,\text{máx}\left\{M,\left|X'(a,\cdot)\right|\right\}+\frac{(b-x)}{2^{1/p}(b-a)}\,\text{máx}\left\{M,\left|X'(b,\cdot)\right|\right\}
\end{aligned}$$

**Theorem 3.12.** *Let $X : [a,b]\times\Omega \to \mathbb{R}$ be a differentiable stochastic process on $(a,b)$ with $a < b$ , $q > 1$ and $X'$ is a mean square integrable on $[a,b]$ . If $|X'|^q$ is a quasi convex stochastic process and $|X'(x,\cdot)|^q \leq M$ then, for $\alpha > 0$ the following inequality holds almost everywhere*

$$\begin{aligned}
&\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)X(x,\cdot)-\frac{\Gamma(\alpha+1)}{b-a}\left((J_{x-}^{\alpha}X)(a,\cdot)+(J_{x+}^{\alpha}X)(b,\cdot)\right)\right|\\
&\leq \frac{(x-a)^{\alpha+1}}{(b-a)(\alpha+1)}.\,\text{máx}\left\{M^{1/q},\left|X'(a,\cdot)\right|\right\}+\frac{(b-x)^{\alpha+1}}{(b-a)(\alpha+1)}\,\text{máx}\left\{M^{1/q},\left|X'(b,\cdot)\right|\right\}
\end{aligned}$$

**Proof.** Using Lemma 3.7, the triangular inequality and Hölder inequality it is had that

$$\begin{aligned}
&\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)X(x,\cdot)-\frac{\Gamma(\alpha+1)}{b-a}\left(\left(J_{x-}^{\alpha}X\right)(a,\cdot)+\left(J_{x+}^{\alpha}X\right)(b,\cdot)\right)\right|\\
&\leq\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left|X'(tx+(1-t)a,\cdot)\right|dt+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left|X'(tx+(1-t)b,\cdot)\right|dt\\
&\leq\frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 t^{\alpha p}dt\right)^{1/p}\left(\int_0^1\left|X'(tx+(1-t)a,\cdot)\right|^q dt\right)^{1/q}\\
&\qquad+\frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 t^{\alpha p}dt\right)^{1/p}\left(\int_0^1\left|X'(tx+(1-t)b,\cdot)\right|^q dt\right)^{1/q}
\end{aligned}$$

where $1/p=1-1/q$.

Since $|X'|$ is a quasi convex stochastic process and the fact that $|X'(x,\cdot)|\leq M$ it is obtained that

$$\begin{aligned}
&\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)X(x,\cdot)-\frac{\Gamma(\alpha+1)}{b-a}\left(\left(J_{x-}^{\alpha}X\right)(a,\cdot)+\left(J_{x+}^{\alpha}X\right)(b,\cdot)\right)\right|\\
&\leq\frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 t^{\alpha p}dt\right)^{1/p}\left(\int_0^1 \text{máx}\left\{\left|X'(x,\cdot)\right|^q,\left|X'(a,\cdot)\right|^q\right\}dt\right)^{1/q}\\
&\qquad+\frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 t^{\alpha p}dt\right)^{1/p}\left(\int_0^1 \text{máx}\left\{\left|X'(x,\cdot)\right|^q,\left|X'(b,\cdot)\right|^q\right\}dt\right)^{1/q}\\
&=\frac{(x-a)^{\alpha+1}}{(b-a)(\alpha+1)}\left(\text{máx}\left\{\left|X'(x,\cdot)\right|^q,\left|X'(a,\cdot)\right|^q\right\}\right)^{1/q}\\
&\qquad+\frac{(b-x)^{\alpha+1}}{(b-a)(\alpha+1)}\left(\text{máx}\left\{\left|X'(x,\cdot)\right|^q,\left|X'(b,\cdot)\right|^q\right\}\right)^{1/q}\\
&=\frac{(x-a)^{\alpha+1}}{(b-a)(\alpha+1)}.\,\text{máx}\left\{M^{1/q},\left|X'(a,\cdot)\right|\right\}+\frac{(b-x)^{\alpha+1}}{(b-a)(\alpha+1)}\,\text{máx}\left\{M^{1/q},\left|X'(b,\cdot)\right|\right\}.
\end{aligned}$$

The proof is complete. ■

**Corollary 3.13.** *Let $X:[a,b]\times\Omega\to\mathbb{R}$ be a differentiable stochastic process on $(a,b)$ with $a<b$ , $q>1$ and $X'$ is a mean square integrable on $[a,b]$ . If $|X'|^q$ is a quasi convex stochastic process and $|X'(x,\cdot)|^q\leq M$ then, for $\alpha>0$ the following inequality holds almost everywhere*

$$\left|X(x,\cdot)-\frac{1}{b-a}\int_a^b X(t,\cdot)dt\right|\leq\frac{(x-a)}{2(b-a)}\,\text{máx}\left\{M^{1/q},\left|X'(a,\cdot)\right|\right\}+\frac{(b-x)}{2(b-a)}\,\text{máx}\left\{M^{1/q},\left|X'(b,\cdot)\right|\right\}.$$

## 4. Conclusion

In the present work were found Riemman-Liouville fractional integral inequalities of Hermite-Hadamard and Ostrowski type involving first and second derivatives of stochastic processes that are quasi convex. Also, from these results were deduced similar inequalities corresponding to the integral of Riemman.

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