Revista MATUA Páginas: 13–28

Facultad de Ciencias Básicas
©Programa de Matemáticas
Vol. V , Nº 1, (2018)

Hermite Hadamard type inequalities for Stochastic Processes whose Second Derivatives are $(m, h_1, h_2)-$Convex using Riemann-Liouville Fractional Integral.

Desigualdades del tipo Hermite-Hadamard para Procesos Estocásticos cuyas Segundas Derivadas son $(m, h_1, h_2)-$Convexas Usando la Integral Fraccional de Riemann-Liouville.

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Abstract

In this work we find new Hermite-Hadamard type inequalities for Stochastic Processes whose second derivatives are $(m, h_1, h_2)-$convex using Riemann-Liouville fractional integral.

Keywords: Hermite-Hadamard Inequality,$(m, h_1, h_2)-$convex Stochastic Processes, Riemann-Liouville Fractional Integral
2015 MSC: 35A23, 60E15, 26A33

Resumen

En el presente trabajo encontramos algunas desigualdades del tipo Hermite-Hadamard para Procesos Estocásticos cuyas segundas derivadas son $(m, h_1, h_2)-$convexos, usando la integral fraccional de Riemann-Liouville.

Palabras claves: Desigualdad de Hermite-Hadamard, Procesos Estocásticos $(m, h_1, h_2)-$Convexos, Integral Fraccional de Riemann-Liouville
2015 MSC: 35A23, 60E15, 26A33

1. Introduction

The study of convex functions has been of interest for mathematical analysis based on the properties that are deduced from this concept. Due to generalization requirements of the convexity concept to obtain new applications, in the last years great efforts have been made in the study and investigation of this topic.

A function $f : I \to \mathbb{R}$ is said to be convex if for all $x, y \in I$ and $t \in [0, 1]$ the inequality

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$$

holds.

Numerous works of investigation have been realized extending results on inequalities for convex functions towards others much more generalized, using new concepts such as $E-$convex ([34]), quasi-convex ([26]), $s-$convex ([3]), logarithmically convex ([1]), $m-$convex ([21]), $h-$convex ([32]), $\phi-$convex ([9]), strongly convex ([13], etc.

A compendium about the history of inequality of Hermite Hadamard can be found in an work of Mirinovic and Lackovic in [18]. The formulation of this result is as follows:

(Hermite-Hadamard Inequality). Let $f : I \to \mathbb{R}$ be a convex fucntion, and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

The inequality of Hermite Hadamard has become a very useful tool in the Theory of Probability and Optimization (See [14])

The study on convex stochastic processes began in 1974 when B. Nagy in [19], applied a characterization of measurable stochastic processes to solving a generalization of the (additive) Cauchy functional equation. In 1980 Nikodem [20] considered convex stochastic processes. In 1995 Skowronski [30] obtained some further results on convex stochastic processes, which generalize some known properties of convex functions. In the year 2014, E. Set et. al. in [25] investigated Hermite-Hadamard type inequalities for stochastic processes in the second sense. For other results related to stochastic processes see [4],[7],[16],[28] , [27], where further references are given.

Fractional calculus has been widely used in the context of inequalities and generalized convexity as observed in the works of Sarikaya et.al. [24] and Liu et.al. [15].

2. Preliminaries

In this section we present some concepts, examples and properties regarding $(m, h_1, h_2)-$ convexity, the calculus for stochastic processes, and some notions of Riemann-Liouville fractional integral as a theoretical framework for the development of this work.

2.1. About $(m, h_1, h_2)-$convexity.

In [2], Alomari M. , Darus M. and Dragomir S.S. introduced the following generalized concept.

Definition 2.1. *Let* $0 < s \leq 1$ *. The function* $f : [0, \infty) \to \mathbb{R}$ *is called a* $s-$*convex function in second sense if*

$$f(tx + (1 - t)y) \leq t^s f(x) + (1 - t)^s f(y) \tag{1}$$

holds for all $x, y \in [0, \infty)$ *and* $t \in [0, 1]$.

A. Barani,S. Barani and S.S.Dragomir, in [5], about Hermite-Hadamard inequalities, introduced the following definition of $P-$convex functions.

Definition 2.2. *We say that a function $f : I \to \mathbb{R}$ is a $P-$convex on I or $f \in P(I)$ if f is non negative and for all $x, y \in I$ and $t \in [0, 1]$ we have*

$$f(tx + (1-t)y) \leq f(x) + f(y) \tag{2}$$

W. Liu, W. Wen and J. Park in [15] introduced the concept of $MT-$convex function.

Definition 2.3. *A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be $MT-$convex function on I , if it is non negative and for all $x, y \in I$ and $t \in (0, 1)$ satisfies the following inequality*

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{3}$$

S. Varošanec in [32], introduced the $h-$convex functions.

Definition 2.4. *Let $h : J \to \mathbb{R}$ be a non negative function, $h \not\equiv 0$, with $(0, 1) \subset J$ and J is an interval of $\mathbb{R}$. A function $f : I \subset \mathbb{R} \to \mathbb{R}$, where I is an interval of $\mathbb{R}$, is said to be $h-$convex function if for all $x, y \in I$ and $t \in [0, 1]$ the following inequality holds*

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y). \tag{4}$$

G. Toader introduced in [31] the concept of $m-$convex function.

Definition 2.5. *For $f : [0, b] \to \mathbb{R}, b > 0$ and $m \in (0, 1]$, if*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y) \tag{5}$$

is valid for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that f is an $m-$convex function.

In [33], B. Xi and F. Qi., introduced the following definition.

Definition 2.6. *Let $h_1, h_2 : [0, 1] \to \mathbb{R}$ and $m \in (0, 1]$. A function $f : [0, \infty) \to \mathbb{R}$ is said to be $(m, h_1, h_2)-$convex function if the inequality*

$$f(tx + m(1-t)y) \leq h_1(t)f(x) + mh_2(t)(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Remark 2.7. *If we choose $m = 1$ and $h_1(t) = t, h_2(t) = 1 - t$, for $t \in [0, 1]$ we obtain the classical definition of convex function. Also, if $m = 1$ and for a fixed $s \in (0, 1]$: $h_1(t) = t^s, h_2(t) = (1-t)^s$ for $t \in [0, 1]$, we have Definition 2.1. If $m = 1$ and $h_1(t) = h_2(t) = 1$, for $t \in [0, 1]$ we obtain Definition 2.2.*

2.2. About Calculus of Stochastic Processes

The following notions corresponds to Stochastic Process and convex Stochastic Process and its generalizations.

Definition 2.8. *Let $(\Omega, \mathcal{A}, P)$ be an arbitrary probability space. A function $X : \Omega \to \mathbb{R}$ is called a random variable if it is $\mathcal{A}$-measurable. Let $(\Omega, \mathcal{A}, P)$ be an arbitrary probability space and let $T \subset \mathbb{R}$ be time. A collection of random variable $X(t, w), t \in T$ with values in $\mathbb{R}$ is called a stochastic processes.*

1. *If $X(t, w)$ takes values in $S = \mathbb{R}^d$ if is called vector-valued stochastic process.*
2. *If the time T can be a discrete subset of $\mathbb{R}$, then $X(t, w)$ is called a discrete time stochastic process.*
3. *If the time T is an interval, $\mathbb{R}^+$ or $\mathbb{R}$, it is called a stochastic process with continuous time.*

Definition 2.9. *Set $(\Omega, \mathcal{A}, P)$ be a probability space and $I \subset \mathbb{R}$ be an interval. We say that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is*

1. *Convex if*
$$X(\lambda u + (1-\lambda)v, \cdot) \leq \lambda X(u, \cdot) + (1-\lambda)X(v, \cdot) \qquad (a.e.) \tag{6}$$
for all $u, v \in I$ and $\lambda \in [0, 1]$.
This class of stochastic process are denoted by C.

Definition 2.10. *Let (Ω, A, P) be a probability space and $I \subset \mathbb{R}$ be an interval. We say that the stochastic process $X : I \times \Omega \to \mathbb{R}$ is called*

1. *Continuous in probability in interval I if for all $t_0 \in I$ we have*
$$P - \lim_{t \to t_0} X(t, \cdot) = X(t_0, \cdot),$$
where $P - lim$ denotes the limit in probability;
2. *Mean-square continuous in the interval I if for all $t_0 \in I$*
$$P - \lim_{t \to t_0} \mathbb{E}(X(t, \cdot) - X(t_0, \cdot)) = 0,$$
where $\mathbb{E}(X(t, \cdot))$ denote the expectation value of the random variable $X(t, \cdot)$;
3. *Increasing (decreasing) if for all $u, v \in I$ such that $t < s$,*
$$X(u, \cdot) \leq X(v, \cdot), \quad (X(u, \cdot) \geq X(v, \cdot)) \qquad (a.e.)$$
4. *Monotonic if it's increasing or decreasing;*
5. *Differentiable at a point $t \in I$ if there is a random variable*
$$X'(t, \cdot) : I \times \Omega \to \mathbb{R}, \text{ such that} X'(t, \cdot) = P - \lim_{t \to t_0} \frac{X(t, \cdot) - X(t_0, \cdot)}{t - t_0}.$$

We say that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is continuous (differentiable) is it is continuous (differentiable) at every point of the interval I (See [16],[12],[30]).

Definition 2.11. *Let (Ω, A, P) be a probability space $T \subset \mathbb{R}$ be an interval with $E(X(t)^2) < \infty$ for all $t \in T$. Let $[a,b] \subset T$, $a = t_0 < t_1 < ... < t_n = b$ be a partition of $[a,b]$ and $\theta_k \in [t_{k-1}, t_k]$ for $k = 1, 2, ..., n$. A random variable $Y : \Omega \to \mathbb{R}$ is called mean-square integral of the process $X(t,\cdot)$ on $[a,b]$ if the following identity holds:*

$$\lim_{n\to\infty} E[X(\theta_k)(t_k - t_{k-1}) - Y(\cdot)]^2 = 0$$

Then we can write

$$\int_a^b X(t,\cdot)dt = Y(\cdot) \qquad (a.e.).$$

Also, mean square integral operator is increasing, that is,

$$\int_a^b X(t,\cdot)dt \leq \int_a^b Z(t,\cdot)dt \qquad (a.e.)$$

where $X(t,\cdot) \leq Z(t,\cdot)$ in $[a,b]$ ([29]).

For other information regarding Stochastic Process Calculation we refer the reader to the following bibliographical references [4, 16].

Now, we give the well-known Hermite-Hadamard integral inequality for convex stochastic processes (see [12]).

Theorem 2.12. *If $X : I \times \Omega \to \mathbb{R}$ is Jensen-convex and mean square continuous in the interval $T \times \Omega$, then for any $u, v \in T$, we have*

$$X\left(\frac{u+v}{2}, \cdot\right) \leq \frac{1}{u-v} \int_u^v X(t,\cdot)dt \leq \frac{X(u,\cdot) + X(v,\cdot)}{2} \qquad (a.e)$$

There is also a generalization of the concept of convexity associated with stochastic processes. In [25] we find the following definition.

Definition 2.13. *Let $0 < s < 1$. A stochastic processes $X : I \times \Omega \to \mathbb{R}$ is said to be $s-$convex stochastic processes in the second sense if*

$$X(ta + (1-t)b, \cdot) \leq t^s X(a,\cdot) + (1-t)^s X(b,\cdot)$$

holds almost everywhere for any $a, b \in I$ and all $t \in [0,1]$.

In a natural way we have

Definition 2.14. *A stochastic processes $X : I \times \Omega \to \mathbb{R}$ is said to be $P-$convex stochastic processes if*

$$X(ta + (1-t)b, \cdot) \leq X(a,\cdot) + X(b,\cdot)$$

holds almost everywhere for any $a, b \in I$ and all $t \in [0,1]$.

The following definitions will be the base for our results.

Definition 2.15. *Let $h_1, h_2 : [0,1] \to \mathbb{R}$ and $m \in (0,1]$. We say that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is a $(m, h_1, h_2)-$convex stochastic process if*

$$X(ta + m(1-t)b, \cdot) \leq h_1(t)X(a,\cdot) + mh_2(t)X(b,\cdot) \qquad (a.e.)$$

for all $a, b \in I$ and $t \in [0,1]$.

Remark 2.16. *We can obtain Definition 2.13 choosing $m = 1$ and for a some fixed $0 < s < 1$: $h_1(t) = t^s, h_2(t) = (1-t)^s$ for $t \in [0,1]$; also if we choose $m = 1$ and $h_1(t) = h_2(t) = 1$, for $t \in [0,1]$ we obtain Definition 2.14*

2.3. About Riemann-Liouville Fractional Integral.

Before we establish our main results, we give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. For more details, one can consult [10, 17, 22].

Definition 2.17. *Let $f \in L_1([a,b])$. The Riemann-Liouville integrals J^{α}_{a+} and J^{α}_{b-} of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt$$

respectively, where $\Gamma(\alpha)$ is the Gamma function and $J^{0}_{a+}f(x) = J^{0}_{b-}f(x) = f(x)$.

Using the Riemann-Liouville fractional integral, Sarikaya et. al. [24], established the Hermite-Hadamard inequalities version.

Theorem 2.18. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $a < b$ and $f \in L_1([a,b])$. If f is a convex function on $[a,b]$ then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left(J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)\right) \leq \frac{f(a)+f(b)}{2}$$

with $\alpha > 0$.

3. Main Results

In this section we will assume that $m \in (0,1]$ and $h_1, h_2 : [0,1] \to \mathbb{R}$ will be non zero functions.

Lemma 3.1. *Let $X : I \times \Omega \to \mathbb{R}$ be a twice differentiable stochastic process on $int(I)$, where $I \subset \mathbb{R}$ is an interval, $a, b \in int(I)$ with $a < b$. If X'' is mean square integrable on $[a,b]$ then the following equality holds almost everywhere:*

$$I(X;\alpha,a,b) = \frac{(b-a)^2}{8}\left[\int_0^1 t^{\alpha+1}X''\left(\frac{t}{2}a + \frac{2-t}{2}b,\cdot\right)dt + \int_0^1 t^{\alpha+1}X''\left(\frac{2-t}{2}a + \frac{t}{2}b,\cdot\right)dt\right] \tag{7}$$

where

$$I(X;\alpha,a,b) = \frac{2^{\alpha-1}\Gamma(\alpha+2)}{(b-a)^{\alpha}}\left(J^{\alpha}_{\frac{a+b}{2}+}X(b,\cdot) + J^{\alpha}_{\frac{a+b}{2}-}X(a,\cdot)\right) - (\alpha+1)X\left(\frac{a+b}{2},\cdot\right).$$

Proof.

Using integration by parts we have

$$\int_0^1 t^{\alpha+1}X''\left(\frac{t}{2}a + \frac{2-t}{2}b,\cdot\right)dt$$

$$
\begin{aligned}
&= \frac{-2}{b-a}\, t^{\alpha+1} X'\left(\frac{t}{2}a + \frac{2-t}{2}b, \cdot\right)\Bigg|_0^1 + \frac{2(\alpha+1)}{a-b}\int_0^1 t^{\alpha} X'\left(\frac{t}{2}a + \frac{2-t}{2}b, \cdot\right)dt \\
&= -\frac{2}{b-a} X'\left(\frac{a+b}{2}, \cdot\right) + \frac{2(\alpha+1)}{b-a}\int_0^1 t^{\alpha} X'\left(\frac{t}{2}a + \frac{2-t}{2}b, \cdot\right)dt
\end{aligned}
$$

and

$$
\int_0^1 t^{\alpha+1} X''\left(\frac{2-t}{2}a + \frac{t}{2}b, \cdot\right)dt
$$

$$
\begin{aligned}
&= \frac{2}{b-a}\, t^{\alpha+1} X'\left(\frac{2-t}{2}a + \frac{t}{2}b, \cdot\right)\Bigg|_0^1 - \frac{2(\alpha+1)}{b-a}\int_0^1 t^{\alpha} X'\left(\frac{2-t}{2}a + \frac{t}{2}b, \cdot\right)dt \\
&= \frac{2}{b-a} X'\left(\frac{a+b}{2}, \cdot\right) - \frac{2(\alpha+1)}{b-a}\int_0^1 t^{\alpha} X'\left(\frac{2-t}{2}a + \frac{t}{2}b, \cdot\right)dt,
\end{aligned}
$$

so, adding these results we have

$$
\begin{aligned}
&\int_0^1 t^{\alpha+1} X''\left(\frac{t}{2}a + \frac{2-t}{2}b, \cdot\right)dt + \int_0^1 t^{\alpha+1} X''\left(\frac{2-t}{2}a + \frac{t}{2}b, \cdot\right)dt \\
&\qquad = \frac{2(\alpha+1)}{b-a}\left[\int_0^1 t^{\alpha} X'\left(\frac{t}{2}a + \frac{2-t}{2}b, \cdot\right)dt - \int_0^1 t^{\alpha} X'\left(\frac{2-t}{2}a + \frac{t}{2}b, \cdot\right)dt\right].
\end{aligned} \tag{8}
$$

Again, using integration by parts and the change of variable $u = \frac{t}{2}a + \frac{2-t}{2}b$ and $v = \frac{2-t}{2}a + \frac{t}{2}b$, we have

$$
\begin{aligned}
\int_0^1 t^{\alpha} X'\left(\frac{t}{2}a + \frac{2-t}{2}b, \cdot\right)dt &= \frac{-2}{b-a}\, t^{\alpha} X\left(\frac{t}{2}a + \frac{2-t}{2}b, \cdot\right)\Bigg|_0^1 + \frac{2\alpha}{b-a}\int_0^1 t^{\alpha-1} X\left(\frac{t}{2}a + \frac{2-t}{2}b, \cdot\right)dt \\
&= \frac{-2}{b-a} X\left(\frac{a+b}{2}, \cdot\right) + \frac{2\alpha}{b-a}\int_0^1 t^{\alpha-1} X\left(\frac{t}{2}a + \frac{2-t}{2}b, \cdot\right)dt
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^1 t^{\alpha} X'\left(\frac{2-t}{2}a + \frac{t}{2}b, \cdot\right)dt &= \frac{2}{b-a} X\left(\frac{a+b}{2}, \cdot\right) - \frac{2\alpha}{b-a}\int_0^1 t^{\alpha-1} X\left(\frac{2-t}{2}a + \frac{t}{2}b, \cdot\right)dt \\
&= \frac{2}{b-a} X\left(\frac{a+b}{2}, \cdot\right) - \frac{2\alpha}{b-a}\int_0^1 t^{\alpha-1} X\left(\frac{2-t}{2}a + \frac{t}{2}b, \cdot\right)dt.
\end{aligned}
$$

Replacing these values in (8), using the definition of the Riemann-Liouville fractional integral and multiplying both sides by $(b-a)^2/8$,we get the desired result (7)

$$
\begin{aligned}
&\frac{(b-a)^2}{8}\left[\int_0^1 t^{\alpha+1} X''\left(\frac{t}{2}a + \frac{2-t}{2}b, \cdot\right)dt + \int_0^1 t^{\alpha+1} X''\left(\frac{2-t}{2}a + \frac{t}{2}b, \cdot\right)dt\right] \\
&\qquad = \frac{2^{\alpha-1}\Gamma(\alpha+2)}{(b-a)^{\alpha}}\left(J_{b-}^{\alpha} X\left(\frac{a+b}{2}, \cdot\right) + J_{a+}^{\alpha} X\left(\frac{a+b}{2}, \cdot\right) - (\alpha+1) X\left(\frac{a+b}{2}, \cdot\right)\right].
\end{aligned}
$$

The proof is complete. ■

Theorem 3.2. *Let $X : I \times \Omega \to \mathbb{R}$ be a twice differentiable stochastic process on $int(I)$, where $I \subset \mathbb{R}$ is an interval, $a, b \in int(I)$ with $a < b$. If $|X''|$ is mean square integrable on $[a, b]$ and (m, h_1, h_2) −convex on $[a, b]$ then*

$$|I(X; \alpha, a, b)| \leq \frac{(b-a)^2}{8(\alpha+2)} \left(K(h_1) \left| X''(a, \cdot) \right| + K(h_2) \left| X''(b, \cdot) \right| \right), \qquad (a.e.) \tag{9}$$

where

$$K(h_1) = \int_0^1 t^{\alpha+1} \left(h_1 \left(\frac{t}{2} \right) + h_1 \left(\frac{2-t}{2} \right) \right) dt$$

and

$$K(h_2) = \int_0^1 t^{\alpha+1} \left(h_2 \left(\frac{t}{2} \right) + h_2 \left(\frac{2-t}{2} \right) \right) dt.$$

Proof. Using Lemma 3.1 we have

$$|I(X; \alpha, a, b)| \leq \frac{(b-a)^2}{8} \left[\left| \int_0^1 t^{\alpha+1} X'' \left(\frac{t}{2} a + \frac{2-t}{2} b, \cdot \right) dt \right| + \left| \int_0^1 t^{\alpha+1} X'' \left(\frac{2-t}{2} a + \frac{t}{2} b, \cdot \right) dt \right| \right]. \tag{10}$$

Since X'' is (m, h_1, h_2) −convex on $[a, b]$ we get that

$$\left| X'' \left(\frac{t}{2} a + \frac{2-t}{2} b, \cdot \right) \right| \leq h_1 \left(\frac{t}{2} \right) \left| X''(a, \cdot) \right| + m h_2 \left(\frac{t}{2} \right) \left| X''(b, \cdot) \right|$$

and

$$\left| X'' \left(\frac{2-t}{2} a + \frac{t}{2} b, \cdot \right) \right| \leq h_1 \left(\frac{2-t}{2} \right) \left| X''(a, \cdot) \right| + m h_2 \left(\frac{2-t}{2} \right) \left| X''(b, \cdot) \right|.$$

So, we can write the inequality (10) as

$$\begin{aligned} |I(X; \alpha, a, b)| &\leq \frac{(b-a)^2}{8} \left[\int_0^1 t^{\alpha+1} h_1 \left(\frac{t}{2} \right) \left| X''(a, \cdot) \right| dt + \int_0^1 t^{\alpha+1} m h_2 \left(\frac{t}{2} \right) \left| X''(b, \cdot) \right| dt \right. \\ &\quad \left. + \int_0^1 t^{\alpha+1} h_1 \left(\frac{2-t}{2} \right) \left| X''(a, \cdot) \right| dt + \int_0^1 t^{\alpha+1} m h_2 \left(\frac{2-t}{2} \right) \left| X''(b, \cdot) \right| dt \right] \\ &\leq \frac{(b-a)^2}{8} \left[\left| X''(a, \cdot) \right| \int_0^1 t^{\alpha+1} \left(h_1 \left(\frac{t}{2} \right) + h_1 \left(\frac{2-t}{2} \right) \right) dt \right. \\ &\quad \left. + m \left| X''(b, \cdot) \right| \int_0^1 t^{\alpha+1} \left(h_2 \left(\frac{t}{2} \right) + h_2 \left(\frac{2-t}{2} \right) \right) dt \right]. \end{aligned}$$

Making

$$K(h_1) = \int_0^1 t^{\alpha+1} \left(h_1 \left(\frac{t}{2} \right) + h_1 \left(\frac{2-t}{2} \right) \right) dt$$

and

$$K(h_2) = \int_0^1 t^{\alpha+1} \left(h_2 \left(\frac{t}{2} \right) + h_2 \left(\frac{2-t}{2} \right) \right) dt,$$

we have the desired result (9).

The proof is complete. ■

Theorem 3.3. *Let $X : I \times \Omega \to \mathbb{R}$ be a twice differentiable stochastic process on int(I), where $I \subset \mathbb{R}$ is an interval, $a, b \in$ int(I) with $a < b$, If $|X''|^q$ is mean square integrable on $[a, b]$ and (m, h_1, h_2) −convex on $[a, b]$ for $q > 1$, then the following inequality holds*

$$|I(X;\alpha,a,b)| \leq \frac{(b-a)^2}{8}M\left[\left(A_1\left|X''\right|^q(a,\cdot)+mA_2\left|X''\right|^q(b,\cdot)\right)^{1/q} + \left(A_3\left|X''\right|^q(a,\cdot)+mA_4\left|X''\right|^q(b,\cdot)\right)^{1/q}\right] \quad (a.e) \tag{11}$$

where $(1/p) + (1/q) = 1$,

$$A_1 = \int_0^1 h_1\left(\frac{t}{2}\right)dt,\ \ A_2 = \int_0^1 h_2\left(\frac{t}{2}\right)dt$$

$$A_3 = \int_0^1 h_1\left(\frac{2-t}{2}\right)dt,\ \ A_4 = \int_0^1 h_2\left(\frac{2-t}{2}\right)dt,$$

and

$$M = \left(\frac{1}{(\alpha+1)p+1}\right)^{1/p}.$$

Proof. From Lemma 3.1 and using the Hölder inequality we have

$$|I(X;\alpha,a,b)| \leq \frac{(b-a)^2}{8}\left(\int_0^1 t^{(\alpha+1)p}\right)^{1/p}\left[\left(\int_0^1\left|X''\left(\frac{t}{2}a+\frac{2-t}{2}b,\cdot\right)\right|^q dt\right)^{1/q} + \left(\int_0^1\left|X''\left(\frac{2-t}{2}a+\frac{t}{2}b,\cdot\right)\right|^q dt\right)^{1/q}\right]. \tag{12}$$

Since $|X|^q$ is (m, h_1, h_2) −convex on $[a, b]$ then

$$\left|X''\right|^q\left(\frac{t}{2}a+\frac{2-t}{2}b,\cdot\right) \leq h_1\left(\frac{t}{2}\right)\left|X''\right|^q(a,\cdot)+mh_2\left(\frac{t}{2}\right)\left|X''\right|^q(b,\cdot)$$

and

$$\left|X''\right|^q\left(\frac{2-t}{2}a+\frac{t}{2}b,\cdot\right) \leq h_1\left(\frac{2-t}{2}\right)\left|X''\right|^q(a,\cdot)+mh_2\left(\frac{2-t}{2}\right)\left|X''\right|^q(b,\cdot).$$

We can write the inequality (12) as

$$\begin{aligned}|I(X;\alpha,a,b)| &\leq \frac{(b-a)^2}{8}\left(\int_0^1 t^{(\alpha+1)p}\right)^{1/p}\left[\left(\int_0^1\left(h_1\left(\frac{t}{2}\right)\left|X''\right|^q(a,\cdot)+mh_2\left(\frac{t}{2}\right)\left|X''\right|^q(b,\cdot)\right)dt\right)^{1/q}\right.\\ &\qquad \left.+\left(\int_0^1\left(h_1\left(\frac{2-t}{2}\right)\left|X''\right|^q(a,\cdot)+mh_2\left(\frac{2-t}{2}\right)\left|X''\right|^q(b,\cdot)\right)dt\right)^{1/q}\right]\\ &= \frac{(b-a)^2}{8}\left(\int_0^1 t^{(\alpha+1)p}\right)^{1/p}\left[\left(\left|X''\right|^q(a,\cdot)\int_0^1 h_1\left(\frac{t}{2}\right)dt+m\left|X''\right|^q(b,\cdot)\int_0^1 h_2\left(\frac{t}{2}\right)dt\right)^{1/q}\right.\\ &\qquad \left.+\left(\left|X''\right|^q(a,\cdot)\int_0^1 h_1\left(\frac{2-t}{2}\right)dt+m\left|X''\right|^q(b,\cdot)\int_0^1 h_2\left(\frac{2-t}{2}\right)dt\right)^{1/q}\right]\end{aligned}$$

doing

$$A_1 = \int_0^1 h_1\left(\frac{t}{2}\right)dt, \;\; A_2 = \int_0^1 h_2\left(\frac{t}{2}\right)dt$$

$$A_3 = \int_0^1 h_1\left(\frac{2-t}{2}\right)dt, \;\; A_4 = \int_0^1 h_2\left(\frac{2-t}{2}\right)dt,$$

and

$$M = \left(\int_0^1 t^{(\alpha+1)p}dt\right)^{1/p} = \frac{1}{(\alpha+1)\,p+1}$$

we get the desired result (11).

The proof is complete. ■

Theorem 3.4. *Let $X : I \times \Omega \to \mathbb{R}$ be a twice differentiable stochastic process on int(I), where $I \subset \mathbb{R}$ is an interval, $a, b \in$ int(I) with $a < b$, If $|X''|^q$ is mean square integrable on $[a, b]$ and (m, h_1, h_2) −convex on $[a, b]$ for $q > 1$, then the following inequality holds*

$$|I(X;\alpha,a,b)| \leq \frac{(b-a)^2}{8}\left[\left(B_1\left|X''\right|^q(a,\cdot) + B_2 m\left|X''\right|^q(b,\cdot)\right)^{1/q} + \left(B_3\left|X''\right|^q(a,\cdot) + B_4 m\left|X''\right|^q(b,\cdot)\right)^{1/q}\right] \quad (a.e.) \tag{13}$$

where

$$B_1 = \int_0^1 t^{\alpha+1}h_1\left(\frac{t}{2}\right)dt, \;\; B_2 = \int_0^1 t^{\alpha+1}h_2\left(\frac{2-t}{2}\right)dt$$

$$B_3 = \int_0^1 t^{\alpha+1}h_1\left(\frac{2-t}{2}\right)dt, \;\; B_4 = \int_0^1 t^{\alpha+1}h_2\left(\frac{t}{2}\right)dt.$$

Proof. From Lemma 3.1 and using the power mean inequality for $q \geq 1$ and the (m, h_1, h_2) −convexity of $|X|^q$ we have

$$\begin{aligned}
|I(X;\alpha,a,b)| &\leq \frac{(b-a)^2}{8}\left[\left|\int_0^1 t^{\alpha+1}X''\left(\frac{t}{2}a+\frac{2-t}{2}b,\cdot\right)dt\right| + \left|\int_0^1 t^{\alpha+1}X''\left(\frac{2-t}{2}a+\frac{t}{2}b,\cdot\right)dt\right|\right] \\
&\leq \frac{(b-a)^2}{8}\left(\int_0^1 t^{\alpha+1}dt\right)^{1-1/q}\left[\left(\int_0^1 t^{\alpha+1}\left|X''\right|^q\left(\frac{t}{2}a+\frac{2-t}{2}b,\cdot\right)dt\right)^{1/q}\right. \\
&\qquad \left. + \left(\int_0^1 t^{\alpha+1}\left|X''\right|^q\left(\frac{2-t}{2}a+\frac{t}{2}b,\cdot\right)dt\right)^{1/q}\right] \\
&\leq \frac{(b-a)^2}{8(\alpha+2)^{1-1/q}}\left[\left(\left|X''\right|^q(a,\cdot)\int_0^1 t^{\alpha+1}h_1\left(\frac{t}{2}\right)dt + m\left|X''\right|^q(b,\cdot)\int_0^1 t^{\alpha+1}h_2\left(\frac{2-t}{2}\right)dt\right)^{1/q}\right. \\
&\qquad \left. + \left(\left|X''\right|^q(a,\cdot)\int_0^1 t^{\alpha+1}h_1\left(\frac{2-t}{2}\right)dt + m\left|X''\right|^q(b,\cdot)\int_0^1 t^{\alpha+1}h_2\left(\frac{t}{2}\right)dt\right)^{1/q}\right]
\end{aligned}$$

Doing

$$B_1 = \int_0^1 t^{\alpha+1}h_1\left(\frac{t}{2}\right)dt, \;\; B_2 = \int_0^1 t^{\alpha+1}h_2\left(\frac{2-t}{2}\right)dt$$

$$B_3=\int_0^1 t^{\alpha+1}h_1\left(\frac{2-t}{2}\right)dt,\quad B_4=\int_0^1 t^{\alpha+1}h_2\left(\frac{t}{2}\right)dt$$

we get the desired result (13).

The proof is complete. ■

4. Some Consequences

Corollary 4.1. *Let $X : I\times\Omega\to\mathbb{R}$ be a twice differentiable stochastic process on int(I), where $I\subset\mathbb{R}$ is an interval, $a,b\in$ int(I) with $a<b$. If $|X''|$ is mean square integrable on $[a,b]$ and convex on $[a,b]$ then*

$$|I(X;\alpha,a,b)|\le\frac{(b-a)^2}{8(\alpha+2)}\left(\frac{|X''(a,\cdot)|}{(\alpha+3)}+\frac{|X''(b,\cdot)|}{(\alpha+1)}\right)\quad (a.e.).\tag{14}$$

Proof. If in Theorem 3.2 we choose $m=1, h_1(t)=t$ and $h_2(t)=1-t$, for all $t\in[0,1]$ then we obtain

$$\begin{aligned}K(h_1) &= \int_0^1 t^{\alpha+1}\left(h_1\left(\frac{t}{2}\right)+h_1\left(\frac{2-t}{2}\right)\right)dt\\ &= \int_0^1 t^{\alpha+2}dt=\frac{1}{\alpha+3}\end{aligned}$$

and

$$\begin{aligned}K(h_2) &= \int_0^1 t^{\alpha+1}\left(h_2\left(\frac{t}{2}\right)+h_2\left(\frac{2-t}{2}\right)\right)dt\\ &= \frac{1}{\alpha+1}.\end{aligned}$$

Making the substitution in the inequality (9) we have the desired result (14).

The proof is complete. ■

Corollary 4.2. *Let $X : I\times\Omega\to\mathbb{R}$ be a twice differentiable stochastic process on int(I), where $I\subset\mathbb{R}$ is an interval, $a,b\in$ int(I) with $a<b$. Let $s\in(0,1]$, if $|X''|$ is mean square integrable on $[a,b]$ and $s-$convex in the second sense on $[a,b]$ then*

$$|I(X;\alpha,a,b)|\le\frac{(b-a)^2}{8}\left(\frac{1}{2^s(s+\alpha+1)}+\frac{2^\alpha\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\right)\left(\left|X''(a,\cdot)\right|+\left|X''(b,\cdot)\right|\right)\quad (a.e.).$$

Proof. Let $s\in(0,1]$. In theorem 3.2 we can choose $m=1$, and the functions $h_1(t)=t^s$ for $t\in(0,1]$, and $h_2(t)=(1-t)^s$ for $t\in(0,1]$. Applying the same technique as in Corollary 4.1 we get desired result. The proof is complete. ■

Corollary 4.3. *Let $X : I\times\Omega\to\mathbb{R}$ be a twice differentiable stochastic process on int(I), where $I\subset\mathbb{R}$ is an interval, $a,b\in$ int(I) with $a<b$. If $|X''|$ is mean square integrable on $[a,b]$ and $P-$convex in the second sense on $[a,b]$ then*

$$|I(X;\alpha,a,b)|\le\frac{(b-a)^2}{4(\alpha+1)}\left(\left|X''(a,\cdot)\right|+\left|X''(b,\cdot)\right|\right)\quad (a.e.).$$

Proof. In theorem 3.2 we can choose $m=1$, and the functions $h_1(t)=h_2(t)=1$ for $t\in(0,1]$. Applying the same technique as in Corollary 4.1 we get desired result. The proof is complete. ■

Remark 4.4. *If we choose $\alpha = 1$ in Corollaries 4.1, 4.2 and 4.3, we have the following inequalities using the Riemann integral, respectively:*

1. *for convex stochastic processes*

$$\left|\frac{1}{(b-a)}\left(\int_a^b X(t,\cdot)dt\right) - X\left(\frac{a+b}{2},\cdot\right)\right| \leq \frac{(b-a)^2}{48}\left(\frac{|X''(a,\cdot)|}{4} + \frac{|X''(b,\cdot)|}{2}\right) \quad (a.e.).$$

2. *for $s-$convex stochastic processes in second sense*

$$\left|\frac{1}{(b-a)}\left(\int_a^b X(t,\cdot)dt\right) - X\left(\frac{a+b}{2},\cdot\right)\right| \leq \frac{(b-a)^2\,\Gamma(s+1)}{4\Gamma(s+4)}\left(\left|X''(a,\cdot)\right| + \left|X''(b,\cdot)\right|\right) \quad (a.e.).$$

3. *for $P-$convex stochastic processes*

$$\left|\frac{1}{(b-a)}\left(\int_a^b X(t,\cdot)dt\right) - X\left(\frac{a+b}{2},\cdot\right)\right| \leq \frac{(b-a)^2}{16}\left(\left|X''(a,\cdot)\right| + \left|X''(b,\cdot)\right|\right) \quad (a.e.).$$

Corollary 4.5. *Let $X : I \times \Omega \to \mathbb{R}$ be a twice differentiable stochastic process on int(I), where $I \subset \mathbb{R}$ is an interval, $a, b \in$ int(I) with $a < b$, If $|X''|^q$ is mean square integrable on $[a, b]$ and convex on $[a, b]$ for $q > 1$, then the following inequality holds almost everywhere*

$$|I(X;\alpha,a,b)| \leq \frac{(b-a)^2}{8} M\left[\left(\frac{|X''|^q(a,\cdot) + 3|X''|^q(b,\cdot)}{4}\right)^{1/q} + \left(\frac{3|X''|^q(a,\cdot) + |X''|^q(b,\cdot)}{4}\right)^{1/q}\right]$$

where $(1/p) + (1/ = 1$ *and*

$$M = \left(\frac{1}{(\alpha+1)\,p+1}\right)^{1/p}.$$

Proof. If in Theorem 3.3 we make $m = 1, h_1(t) = t$ and $h_2(t) = 1 - t$ then we have

$$A_1 = \int_0^1 h_1\left(\frac{t}{2}\right)dt = \int_0^1 \frac{t}{2}dt = \frac{1}{4} = A_4$$

$$A_2 = \int_0^1 h_2\left(\frac{t}{2}\right)dt = \int_0^1 \left(1 - \frac{t}{2}\right)dt = \frac{3}{4} = A_3$$

then the inequality for convex Stochastic Processes has the form

$$|I(X;\alpha,a,b)| \leq \frac{(b-a)^2}{8} M\left[\left(\frac{|X''|^q(a,\cdot) + 3|X''|^q(b,\cdot)}{4}\right)^{1/q} + \left(\frac{3|X''|^q(a,\cdot) + |X''|^q(b,\cdot)}{4}\right)^{1/q}\right].$$

■

Corollary 4.6. *Let $X : I \times \Omega \to \mathbb{R}$ be a twice differentiable stochastic process on int(I), where $I \subset \mathbb{R}$ is an interval, $a, b \in$ int(I) with $a < b$. Let $s \in (0, 1]$, if $|X''|^q$ is mean square integrable on $[a, b]$ and $s-$convex in the second sense on $[a, b]$ for $q > 1$, then the following inequality holds almost everywhere*

$$|I(X;\alpha,a,b)| \leq \frac{(b-a)^2}{2^{3+s/q}(s+1)^{1/q}} M\left[\left(\left|X''\right|^q(a,\cdot) + \left(2^{s+1} - 1\right)\left|X''\right|^q(b,\cdot)\right)^{1/q}\right.$$

$$+\left(\left(2^{s+1}-1\right)\left|X''\right|^{q}(a,\cdot)+\left|X''\right|^{q}(b,\cdot)\right)^{1/q}\Bigg]$$

where $(1/p)+(1/q)=1$ *and*

$$M=\left(\frac{1}{(\alpha+1)\,p+1}\right)^{1/p}.$$

Proof. Let $s\in(0,1]$. In theorem 3.3 we can choose $m=1$, and the functions $h_1(t)=t^s$ for $t\in(0,1]$, and $h_2(t)=(1-t)^s$ for $t\in(0,1]$. Applying the same technique as in Corollary 4.5 we get desired result. The proof is complete. ■

Corollary 4.7. *Let* $X: I\times\Omega\to\mathbb{R}$ *be a twice differentiable stochastic process on* $int(I)$, *where* $I\subset\mathbb{R}$ *is an interval,* $a,b\in int(I)$ *with* $a<b$. *Let* $s\in(0,1]$, *if* $|X''|^q$ *is mean square integrable on* $[a,b]$ *and* $P-$*convex in the second sense on* $[a,b]$ *for* $q>1$, *then the following inequality holds almost everywhere*

$$|I(X;\alpha,a,b)|\le\frac{(b-a)^2}{4}M\left[\left(\left|X''\right|^{q}(a,\cdot)+\left|X''\right|^{q}(b,\cdot)\right)^{1/q}\right],$$

where $(1/p)+(1/q)=1$ *and*

$$M=\left(\frac{1}{(\alpha+1)\,p+1}\right)^{1/p}.$$

Proof. In theorem 3.3 we can choose $m=1$, and the functions $h_1(t)=h_2(t)=1$ for $t\in(0,1]$. Applying the same technique as in Corollary 4.1 we get desired result. The proof is complete. ■

Remark 4.8. *If we choose* $\alpha=1$ *in Corollaries 4.5, 4.6 and 4.7 then we get the following inequalities using the Riemann integral, respectively, almost everywhere:*

1. *for convex stochastic processes*
$$\left|\frac{1}{(b-a)}\left(\int_a^b X(t,\cdot)dt\right)-X\left(\frac{a+b}{2},\cdot\right)\right|$$
$$\le\frac{(b-a)^2}{16\left((2p+1)^{1/p}\right.}\left[\left(\frac{|X''|^q(a,\cdot)+3\,|X''|^q(b,\cdot)}{4}\right)^{1/q}+\left(\frac{3\,|X''|^q(a,\cdot)+|X''|^q(b,\cdot)}{4}\right)^{1/q}\right].$$
2. *for* $s-$*convex stochastic processes in the second sense*
$$\left|\frac{1}{(b-a)}\left(\int_a^b X(t,\cdot)dt\right)-X\left(\frac{a+b}{2},\cdot\right)\right|\le$$
$$\frac{(b-a)^2}{2^{4+s/q}(s+1)^{1/q}(2p+1)^{1/p}}\left[\left(\left|X''\right|^{q}(a,\cdot)+\left(2^{s+1}-1\right)\left|X''\right|^{q}(b,\cdot)\right)^{1/q}\right.$$
$$\left.+\left(\left(2^{s+1}-1\right)\left|X''\right|^{q}(a,\cdot)+\left|X''\right|^{q}(b,\cdot)\right)^{1/q}\right]$$
3. *for* $P-$*convex stochastic processes*
$$\left|\frac{1}{(b-a)}\left(\int_a^b X(t,\cdot)dt\right)-X\left(\frac{a+b}{2},\cdot\right)\right|\le\frac{(b-a)^2}{4(2p+1)^{1/p}}\left[\left(\left|X''\right|^{q}(a,\cdot)+\left|X''\right|^{q}(b,\cdot)\right)^{1/q}\right].$$

Corollary 4.9. *Let $X : I \times \Omega \to \mathbb{R}$ be a twice differentiable stochastic process on $int(I)$, where $I \subset \mathbb{R}$ is an interval, $a, b \in int(I)$ with $a < b$, If $|X''|^q$ is mean square integrable on $[a, b]$ and convex on $[a, b]$ for $q > 1$, then the following inequality holds almost everywhere*

$$|I(X; \alpha, a, b)| \leq \frac{(b-a)^2}{2^{3+1/q}(\alpha+3)}\left[\left(\left|X''\right|^q(a,\cdot) + \frac{|X''|^q(b,\cdot)}{(\alpha+2)}\right)^{1/q} + \left(\frac{|X''|^q(a,\cdot)}{(\alpha+2)} + \left|X''\right|^q(b,\cdot)\right)^{1/q}\right].$$

Proof. If in Theorem 3.4 we make $m = 1, h_1(t) = h_2(t) = t$ then we have

$$B_1 = \int_0^1 t^{\alpha+1} h_1\left(\frac{t}{2}\right)dt = \int_0^1 t^{\alpha+1}\frac{t}{2}dt = \frac{1}{2(\alpha+3)} = B_4$$

$$B_2 = \int_0^1 t^{\alpha+1} h_2\left(\frac{2-t}{2}\right)dt = \int_0^1 t^{\alpha+1}\frac{2-t}{2}dt = \left(\frac{1}{2(\alpha+2)} - \frac{1}{2(\alpha+3)}\right) = B_3$$

so, the inequality for convex stochastic processes has the form

$$|I(X; \alpha, a, b)| \leq \frac{(b-a)^2}{2^{3+1/q}(\alpha+3)}\left[\left(\left|X''\right|^q(a,\cdot) + \frac{|X''|^q(b,\cdot)}{(\alpha+2)}\right)^{1/q} + \left(\frac{|X''|^q(a,\cdot)}{(\alpha+2)} + \left|X''\right|^q(b,\cdot)\right)^{1/q}\right].$$

■

Remark 4.10. *If we choose $\alpha = 1$ in Corollary (4.9), then we get the following inequality using the Riemann integral:*

$$\left|\frac{2}{(b-a)}\left(\int_a^b X(t,\cdot)dt\right) - 2X\left(\frac{a+b}{2},\cdot\right)\right| \leq \frac{(b-a)^2}{2^{5*1/q}}\left[\left(\left|X''\right|^q(a,\cdot) + \frac{|X''|^q(b,\cdot)}{3}\right)^{1/q} + \left(\frac{|X''|^q(a,\cdot)}{3} + \left|X''\right|^q(b,\cdot)\right)^{1/q}\right]$$

almost everywhere; in addition, if $q = 1$ then

$$\left|\frac{2}{(b-a)}\left(\int_a^b X(t,\cdot)dt\right) - 2X\left(\frac{a+b}{2},\cdot\right)\right| \leq \frac{(b-a)^2}{48}\left(\left|X''\right|(a,\cdot) + \left|X''\right|(b,\cdot)\right) \qquad (a.e.).$$

5. Conclusion

In the development of this work we have found some new inequalities of the Hermite-Hadamard type valid for stochastic processes whose second derivatives are $(m, h_1, h_2)-$ convex, using the fractional integral of Riemann -Liouville. Additionally, as corollaries and observations, we have achieved some consequences derived from these results. We hope that this work will be motivating for new research in this area.

Aknowledgment

The authors would like to thank the Council of Scientific, Humanistic and Technological Development (Consejo de Deasarrollo Científico, Humanístico y Tecnológico) of the Centroccidental University Lisandro Alvarado (Universidad Centroccidental Lisandro Alvarado, Venezuela) for the technical support provided in the preparation of this work (Project RAC-2018-01), as well as the arbitrators appointed for the evaluation of this work and the editorial body of the Matua Journal of the Mathematics Program of Atlantic University (Revista del Programa de Matemáticas de la Universidad del Atlántico, Colombia).

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