

Sobre la Existencia y Unicidad de Soluciones para un Sistema de Ecuaciones No Lineal.

On the Existence and Uniqueness of the Solutions of a System of Non-Linear Differential Equations.

Miguel José Vivas Cortez^a, Juan C. Osorio^b

mjvivas@puce.edu.ec and jcosorio@puce.edu.ec

^a*Pontificia Universidad Católica del Ecuador*

Facultad de Ciencias Naturales y Exactas

Escuela de Ciencias Físicas y Matemáticas, Sede Quito, Ecuador

^b*Pontificia Universidad Católica del Ecuador. (PUCE)*

Resumen

En este artículo demostraremos un teorema de existencia y unicidad para un sistema de ecuaciones lineales que incluye como consecuencia a los Modelos de Volterra

Palabras claves: Sistemas No Lineales, Ecuaciones diferenciales, Modelos de Volterra.

Abstract

In this work we present an existence and uniqueness Theorem for a very especial class of a non-linear system of differential equations which include The Volterra Models..

Keywords: non-linear systems, differential equations, Volterra Models.

1. Introduction

The study of differential equations have multiple impact in science and everyday life. It is the case that the Lotka-Volterra type of models are the most people work on, but also press-predator models and competitive models as well [1][2].

It is also well know the extraordinary development of the theory of differential equations with finite or infinite delay.

In [1], Montes de Oca and Miguel Vivas, studied the system of differential of Lotka-Volterra type with

infinite delay.

$$\begin{cases} x_1'(t) = x_1(t)[a(t) - b(t)x_1(t) - c_1(t) \int_{-\infty}^t k_1(t-s)x_2(s) ds] \\ x_2'(t) = x_2(t)[d(t) - f(t)x_2(t) - e_1(t) \int_{-\infty}^t k_2(t-s)x_1(s) ds] & \text{if } t \geq t_0 \\ x_1(t) = \phi_1(t) \wedge x_2(t) = \phi_2(t) & \text{if } t < t_0 \end{cases} \quad (1)$$

where the derivation at t_0 should be interpreted as the derivative from the right, that is to say; $x_i'(t_0) = x_{i+}'(t_0)$ for $i = 1, 2$ and $a(t), b(t), c(t), d(t), e(t)$, y $f(t)$ are bounded positive from above and from below with positive constants which satisfies

$$c(t)d(t) \leq a(t)f(t)$$

$$b(t)d(t) \leq (t)a(t)$$

And also $k_i : [0, +\infty) \rightarrow [0, +\infty)$ are continuous and positive kernels such that: $\int_0^{+\infty} k_i(s) ds = 1$ and the ϕ_i 's are the initial conditions.

In this work we present an existence and uniqueness theorem for the more general case than (1), System (1) can be written as:

$$x'(t) - h(t, x(t)) - A(x(t)) \int_{-\infty}^t g(t, \tau, x(\tau)) d\tau \quad (2)$$

where

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}; A(x) = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$$

$$h : (-\infty, +\infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$g : \{(t, s) \in \mathbb{R}^2 / s \leq t\} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

are continuous and are given by

$$h(t, x) = \begin{pmatrix} a(t)x_1 - b(t)(x_1)^2 \\ d(t)x_2 - e(t)(x_2)^2 \end{pmatrix} \quad (3)$$

$$g(t, s, x) = \begin{pmatrix} c(t)k_1(t-s)x_2 \\ f(t)k_2(t-s)x_1 \end{pmatrix} \quad (4)$$

It is very important to note that:

$$\|A(t)\| \leq |x_1| + |x_2| = \|x\| \quad (*)$$

1.1. Preliminary Results

Lemma 1. The function $h : (-\infty, +\infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by (3) satisfied a local Lipschitz condition on x , in the sense that for each $(t_0, x_0) \in (\mathbb{R}^* \cup \{0\}) \times \mathbb{R}^2$ and each $M > 0$, There exist $k > 0$ such that, if $\|x - x_0\| \leq M$; $\|\bar{x} - x_0\| \leq M$ and $|t - t_0| \leq M$, then;

$$\|h(t, x) - h(t, \bar{x})\| \leq k\|x - \bar{x}\|$$

where

$$\|(x_1, x_2)\| = |x_1| + |x_2|$$

Proof:

Let us observe that

$$k = \max\{a_M + 2Mb_M, d_M + 2Me_M\}$$

satisfies the previous conditions; where

$$a_L \leq a(t) \leq a_M, \quad b_L \leq b(t) \leq b_M$$

$$d_L \leq d(t) \leq d_M \quad \text{and} \quad e_L \leq e(t) \leq e_M.$$

Lemma 2. The function g defined in (4) satisfies a local Lipschitz condition on x , in the following sense:
For each $M > 0$ There exist $k > 0$ such that:
if $-M \leq s \leq t \leq M$, $\|x\| \leq M$, $\|\bar{x}\| \leq M$ Then;

$$\|g(t, s, x) - g(t, s, \bar{x})\| \leq k\|x - \bar{x}\|$$

Proof:

$$\begin{aligned} \|g(t, s, x) - g(t, s, \bar{x})\| &= \left\| \begin{pmatrix} c(t)k_1(t-s)(x_2 - \bar{x}_2) \\ f(t)k_2(t-s)(x_1 - \bar{x}_1) \end{pmatrix} \right\| \\ &\leq k(|x_1 - \bar{x}_1| + |x_2 - \bar{x}_2|) \\ &= k\|x - \bar{x}\| \\ k &= \max\{c_M \max_{s \in [0, M]} k_1(s), c_M \max_{s \in [0, M]} k_2(s)\} \end{aligned}$$

Lemma 3. let $t_0 \geq 0$, then the set

$$B(t_0) = \{\Phi = (\phi_1, \phi_2) \mid \phi_1, \phi_2 \in FA_{t_0}\}$$

where

$$FA_{t_0} = \{\phi : (-\infty, t_0) \longrightarrow \mathbb{R}^+ \cup \{0\}) \mid \phi \text{ is continuous, bounded above and } \phi(t_0) > 0\}$$

is a convex subset of the space of all continuous functions $\Phi : (-\infty, t_0] \longrightarrow \mathbb{R}^2$ and satisfies that if $\Phi \in B(t_0)$, then

$$G(t, t_0, \Phi) = \int_{-\infty}^{t_0} g(t, s, \Phi(s)) ds$$

defines a continuous function in $[t_0, +\infty)$ where

$$g(t, s, (\phi_1, \phi_2)) = \begin{pmatrix} c(t)k_1(t-s)\phi_2 \\ f(t)k_2(t-s)\phi_1 \end{pmatrix}$$

Proof:

let $\alpha, \lambda \in [0, 1]$ with $\alpha + \lambda = 1$ and $(\phi_1, \phi_2), (\bar{\phi}_1, \bar{\phi}_2) \in B(t_0)$. Then

$$\alpha(\phi_1, \phi_2) + \lambda(\bar{\phi}_1, \bar{\phi}_2) = (\alpha\phi_1 + \lambda\bar{\phi}_1, \alpha\phi_2 + \lambda\bar{\phi}_2)$$

Also, $\phi_1, \bar{\phi}_1, \phi_2, \bar{\phi}_2 \in FA_{t_0}$ and α, λ are non-negative and they do not vanish simultaneously, then $\alpha\phi_1, \lambda\bar{\phi}_1, \alpha\phi_2, \lambda\bar{\phi}_2$ are continuous on $[t_0, +\infty)$, non-negative, strictly positive on to and bounded.

Thus $\alpha\phi_1 + \lambda\bar{\phi}_1, \alpha\phi_2 + \lambda\bar{\phi}_2$ belong to FA_{t_0} . from where $(\alpha\phi_1 + \lambda\bar{\phi}_1, \alpha\phi_2 + \lambda\bar{\phi}_2) \in B(t_0)$ and therefore $B(t_0)$ is convex.

On the other hand,

i.) $G(t, t_0, \Phi)$ is well defined since

$$\begin{aligned} \int_{-\infty}^{t_0} c(t)k_1(t-s)\phi_2(s) ds &\leq c_M\phi_{2M} \int_{-\infty}^{t_0} k_1(t-s) ds \\ &= c_M\phi_{2M} \int_{t-t_0}^{+\infty} k_1(\sigma) d\sigma \\ &\leq c_M\phi_{2M} \int_0^{+\infty} k_1(\sigma) d\sigma \\ &= c_M\phi_{2M} < +\infty \end{aligned}$$

similarly

$$\int_{-\infty}^{t_0} f(t)k_2(t-s)\phi_1(s) ds \leq f_M\phi_{1M} < +\infty$$

ii.) $G(t, t_0, \Phi)$ is continuous at each $t_0 \leq \bar{t}$, being that if we consider the interval $[t_0, d]$ such that $t_0 \leq \bar{t} \leq d$, then each component

$$G_1(t, t_0, \Phi) = G_1(t, t_0, (\phi_1, \phi_2)) = \int_{-\infty}^{t_0} c(t)k_1(t-s)\phi_2(s) ds$$

and

$$G_2(t, t_0, \Phi) = \int_{-\infty}^{t_0} f(t)k_2(t-s)\phi_1(s) ds$$

is continuous in \bar{t} . to see this look at Theorem 14-21, Page 421 [4]. Now, the integral

$$\int_{-\infty}^{t_0} c(t)k_1(t-s)\phi_2(s) ds$$

can be written as

$$\int_{-t_0}^{+\infty} c(t)k_1(t+s)\phi_2(-s) ds$$

Now

$$\begin{aligned} \int_{-t_0}^{+\infty} c(t)\phi_2(s)k_1(t+s) ds &\leq c_M\Phi_{2M} \int_{-t_0}^{+\infty} k_1(t+s) ds \\ &\leq c_M\Phi_{2M} \int_{t-t_0}^{+\infty} k_1(\sigma) d\sigma \\ &\leq c_M\Phi_{2M}\bar{k}_1 \end{aligned}$$

Also, given $\varepsilon > 0$ there exists $R > 0$ such that

$$\left| \int_{-t_0}^b k_1(\sigma) d\sigma - \int_{-t_0}^{+\infty} k_1(\sigma) d\sigma \right| < \frac{\varepsilon}{c_M\Phi_{2M} + 1}$$

$\forall b \geq R$ and $t \in [t_0, d]$ let $b > R - t_0$, then $t_0 \leq t \leq d$ implies that $t_0 + b \leq t + b \leq d + b$ which also implies that $R \leq t + b \leq d + b$ son, given $\varepsilon > 0$ there exists $R > 0$ such that for $b > R - t_0$

$$\begin{aligned} \int_b^{+\infty} c(t)\phi_2(s)k_1(t+s)ds &\leq c_M\Phi_{2M} \int_b^{+\infty} k_1(t+s)ds \\ &= c_M\Phi_{2M} \int_{t+b}^{+\infty} k_1(\sigma)d\sigma \\ &\leq c_M\Phi_{2M}\varepsilon < \varepsilon \end{aligned}$$

for every $t \in [t_0, d]$ Therefore, $\int_b^{+\infty} c(t)\phi_2(s)k_1(t+s)ds$ converges uniformly on $[t_0, d]$. In consequence $G_1(t, t_0, \Phi)$ is continuous in \bar{t} (we say even more in $[t_0, d]$). By the same token $G_2(t, t_0, \Phi)$ is continuous in \bar{t} . Therefore $G(t, t_0, \Phi)$ is continuous for every $\bar{t} \geq t_0$.

2. Existence and Uniqueness Theorem

An existence and theorem or equation (2) is given.

In fact, given a real number and an initial function $\Phi \in B(t_0)$ we look for a continuous solution $x(t) = x(t, t_0, \Phi)$ that satisfies (2) for every $t \in [t_0, t_0 + \beta]$ for some $\beta > 0$ and $x(t) = \phi(t)$ for all $t \leq t_0$. let us observe that if $x(t)$ is a solution, that $x(t)$ is also a solution of the integral equation.

$$x(t) = \begin{cases} \phi(t) & \text{if } t \leq t_0 \\ f(t) & \text{if } t \in [t_0, t_0 + \beta] \end{cases} \quad (I)$$

where

$$f(t) = \Phi(t_0) + \int_{t_0}^t [h(s, x(s)) - A(x(s))G(s, t_0, \Phi)]ds - \int_{t_0}^t \int_{t_0}^s [A(x(s))g(s, \tau, x(\tau))]d\tau ds$$

Conversely, every function $x(t)$ which satisfies (I) is necessarily a solution of the system (2) with initial function Φ . So the problem of existence of solutions of equation (2) is equivalent to the problem of existence of (I).

The right hand side of (I) define a continuous function in $(-\infty, t_0 + \beta)$ for every $\Phi \in B(t_0)$, $t \geq 0$ and

$$x(t) = \begin{cases} \phi(t) & \text{if } t \leq t_0 \\ \omega(x_1, x_2) & \text{if } t_0 \leq t \leq t_0 + \beta \end{cases}$$

even more, it is continuously differentiable on $[t_0, t_0 + \beta]$.

From that point of view, (I) allows us to define an operator P which send the continuous function

$$x(t) = \begin{cases} \phi(t) & \text{if } t \leq t_0 \\ \omega(x_1, x_2) & \text{if } t_0 \leq t \leq t_0 + \beta \end{cases}$$

to the continuous function given by the right hand side of (I).

In particular, if $x(t) = x(t, t_0, \Phi)$ is a solution of I , then,

$$(Px)(t) = x(t)$$

Then, the solutions to (I), or it is equivalent in the original problem (2) with initial condition (t_0, Φ) , appear to be the fixed points of the operator P .

Now, to determinate the existence and uniqueness of a fixed point we will use the principle of the contraction applications.

For that, we need to define P on a subset S of the continuous function from $(-\infty, t_0 + \beta] \rightarrow \mathbb{R}^2$ to which we impose certain conditions such that it is a complete metric space, P is a map from S to itself and P and P be a contraction.

Given problem (2) with the initial condition $\Phi \in B(t_0)$, $t_0 \geq 0$, and positive constants M and β , we consider the set S those function x which satisfies the following conditions:

- a.) $x \in C((-\infty, t_0 + \beta], \mathbb{R}^2)$
- b.) $x(t) = \phi(t)$ if $t \leq t_0$
- c.) $\|x(t_1) - x(t_2)\| \leq M|t_1 - t_2|$ for $t_1, t_2 \in [t_0, t_0 + \beta]$
- d.) $\|x(t) - \Phi(t_0)\| \leq 1$ for $t \in [t_0, t_0 + \beta]$

and then on S we define the function.

$$\rho(x_1, x_2) = \max_{t \in [t_0, t_0 + \beta]} \|x_1(t) - x_2(t)\|$$

So (S, ρ) is a complete metric space, and if M and β are selected appropriately, P will be a contraction from S to itself.

Lemma 4. *There exist M and β such that $\Phi \in B(t_0)$, $t_0 \leq 0$ such that the operator P act from (S, ρ) to itself and is a contraction*

Proof:

Let β_1 , be a positive real number less than 1, now because of the continuity of the functions h, g, G and the compactness of the interval $[t_0, t_0 + \beta]$ and the sets

$$C_1 = \{(t, x) \in \mathbb{R} \times \mathbb{R}^2 / \|x(t) - \Phi(t_0)\| \leq 1, t \in [t_0, t_0 + \beta_1]\}$$

$$C_2 = \{(t, s, x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 / \|x(t) - \Phi(t_0)\| \leq 1, t_0 \leq s \leq t \leq t_0 + \beta_1\}$$

there exists positive constants M_1, M_2 and M_3 such that

$$\|h(t, x)\| \leq M_1 \quad \text{for all } (t, x) \in C_1 \quad (5)$$

$$\|G(t, t_0, \Phi)\| \leq M_2 \quad \text{for all } t \in [t_0, t_0 + \beta] \quad (6)$$

$$\|g(s, \tau, x)\| \leq M_3 \quad \text{for all } (s, \tau, x) \in C_3 \quad (7)$$

We choose $M \in \mathbb{R}^+$ such that

$$M_1 + (M_2 M_3)(1 + \|a(\Phi(t_0))\|) \leq M \quad (8)$$

If we apply the local Lipschitz condition for $h(t, x)$ at the point $(t_0, \Phi(t_0))$ to the set

$$C_3 = \{(t, x) \in \mathbb{R} \times \mathbb{R}^2 / \|x(t) - \Phi(t_0)\| \leq 1, |t - t_0| \leq 1\}$$

we obtain that there exist $L_1 \in \mathbb{R}^+$ such that if (t, x) and $(t, \bar{x}) \in C_3$ then

$$\|h(t, x) - h(t, \bar{x})\| \leq L_1|x - \bar{x}| \quad (9)$$

Similarly, using the Lipschitz condition for g in the set

$$C_4 = \{(t, s, x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 / \|x\| \leq T, -T \leq s \leq t \leq T\}$$

where $T = \max\{t_0 + \beta, 1 + \|\Phi(t_0)\|\}$, there exist an $L_2 \in \mathbb{R}^+$ such that if $(t, s, x), (t, s, \bar{x}) \in C_4$ then

$$\|g(t, s, x) - g(t, s, \bar{x})\| \leq L_2 \|x - \bar{x}\| \quad (10)$$

Let $L = \max\{L_1, L_2\}$ and pick $\beta \in \mathbb{R}^+$ such that

$$\beta < \beta_1 \quad \text{and} \quad \beta_1 < \frac{1}{2M + 2L + L\|\Phi(t_0)\|} \quad (11)$$

The condition $0 < \beta < \beta_1 < 1$ implies that $\beta^2 < \beta$. The second condition implies that $\beta < \frac{1}{M}$. If we consider the space (S, ρ) with M by (8) and β given by (11) then for $x \in S$ we have that.

$$\|x(s) - \Phi(t_0)\| \leq 1$$

for every $s \in [t_0, t_0 + \beta]$
therefore

$$(s, x(s)) \in C_1 \quad \text{for every} \quad s \in [t_0, t_0 + \beta] \quad (12)$$

It is also easy to verify that if $t_0 \leq \tau \leq s \leq t \leq t_0 + \beta$ then $(s, \tau, x(\tau)) \in C_2$ and the

$$\beta_1 \|g(s, \tau, x(\tau))\| \leq M_3 \quad \text{for} \quad t_0 \leq \tau \leq s \leq t \leq t_0 + \beta \quad (13)$$

Also, if $x, \bar{x} \in S$ then $(s, x(s))$ and $(s, \bar{x}(s)) \in C_3$ for every $s \in [t_0, t_0 + \beta]$ and $(s, \tau, x(\tau)), (s, \tau, \bar{x}(\tau)) \in C_4$ if $t_0 \leq \tau \leq s \leq t \leq t_0 + \beta$.

So we get that

$$\|h(s, x(s)) - h(s, \bar{x}(s))\| \leq L \|x(s) - \bar{x}(s)\| \leq L \rho(x, \bar{x}) \quad s \in [t_0, t_0 + \beta] \quad (14)$$

And every $t_0 \leq \tau \leq s \leq t \leq t_0 + \beta$

$$\|g(s, \tau, x(\tau)) - g(s, \tau, \bar{x}(\tau))\| \leq L \|x(s) - \bar{x}(s)\| \leq L \rho(x, \bar{x}). \quad (15)$$

The technique we will use in the next proof is similar to that of Theorem 3.3.5 Page 193 [3].

2.1. The Main Theorem

Theorem 1. (Existence and Uniqueness) Let $t_0 \geq 0$ and $\Phi \in B(t_0)$. Then there exist a unique solution $x(t) = x(t, t_0, \Phi)$ of (2) defined in the interval $[t_0, t_0 + \beta]$ for some $\beta > 0$ and $x(t, t_0, \Phi) = \Phi(t)$ for $x \leq t_0$.

Proof:

Consider the metric space (S, ρ) given above, with M as in (8) and β as in (11).

Let the operator P , define for $x \in S$ by

$$(Px)(t) = \begin{cases} \Phi(t) & \text{if } t \geq t_0 \\ \Phi(t_0) + \int_{t_0}^t [h(s, x(s)) - A(x(s))G(s, t_0, \Phi)] ds \\ - \int_{t_0}^t \int_{t_0}^s [A(x(s))g(s, \tau, x(\tau))] d\tau ds & \text{if } t \in [t_0, t_0 + \beta] \end{cases}$$

from (5), (6), (8), (12) and (13) P is a mapping from S to S and

$$\begin{aligned}
 \|Px(t) - \Phi(t_0)\| &\leq \int_{t_0}^t \|h(s, x(s))\| ds + \int_{t_0}^t \|A(x(s))\| \|G(s, t_0, \Phi)\| ds \\
 &\quad + \int_{t_0}^t \|A(x(s))\| \int_{t_0}^s \|g(s, \tau, x(\tau))\| d\tau ds \\
 &\leq M_1 \int_{t_0}^t ds + \int_{t_0}^t \|x(s)\| M_2 ds + \int_{t_0}^t \|x(s)\| \int_{t_0}^s \frac{M_3}{\beta_1} d\tau ds \\
 &\leq M_1(t - t_0) + (1 + \|\Phi(t_0)\|) M_2(t - t_0) \\
 &\quad + (1 + \|\Phi(t_0)\|) \frac{M_3}{2} \frac{(t - t_0)^2}{\beta_1} \\
 &\leq M_1\beta + (1 + \|\Phi(t_0)\|) M_2\beta + (1 + \|\Phi(t_0)\|) \frac{M_3}{2} \frac{\beta^2}{\beta_1}
 \end{aligned}$$

Then

$$\begin{aligned}
 \|Px(t) - \Phi(t_0)\| &\leq M_1\beta + (1 + \|\Phi(t_0)\|) M_2\beta + (1 + \|\Phi(t_0)\|) M_3\beta \\
 &= (M_1 + (1 + \|\Phi(t_0)\|)(M_2 + M_3))\beta \leq M\beta < 1
 \end{aligned}$$

Similar, for $t_1, t_2 \in [t_0, t_0 + \beta]$ with $t_1 < t_2$ we have that

$$\begin{aligned}
 \|(Px)(t_1) - (Px)(t_2)\| &\leq \int_{t_1}^{t_2} \|h(s, x(s))\| ds + \int_{t_1}^{t_2} \|A(x(s))\| \|G(s, t_0, \Phi)\| ds \\
 &\quad + \int_{t_1}^{t_2} \|A(x(s))\| \int_{t_0}^s \|g(s, \tau, x(\tau))\| d\tau ds \\
 &\leq M_1 \int_{t_1}^{t_2} ds + (1 + \|\Phi(t_0)\|) M_2 \int_{t_0}^{t_2} ds \\
 &\quad + \int_{t_0}^{t_2} (1 + \|\Phi(t_0)\|) \int_{t_0}^s \frac{M_3}{\beta} d\tau ds \\
 &\leq M_1(t_2 - t_1) + (1 + \|\Phi(t_0)\|) M_2(t_2 - t_1) \\
 &\quad + (1 + \|\Phi(t_0)\|) \frac{M_3}{\beta} \int_{t_1}^{t_2} (s - t_0) ds \\
 &\leq M(t_2 - t_1) + (1 + \|\Phi(t_0)\|) M(t_2 - t_1) \\
 &\quad + (1 + \|\Phi(t_0)\|) \frac{M_3}{\beta} (t_2 - t_1) \\
 &= (M_1 + (1 + \|\Phi(t_0)\|)(M_2 + M_3))(t_2 - t_1) \leq M(t_2 - t_1)
 \end{aligned}$$

So Px satisfies the conditions (a),(b),(c), and (d), that is to say that $(Px) \in S$ Finally, let $x, \bar{x} \in S$ and $t \in [t_0, t_0 + \beta]$, now, from (5), (6), (12), (13) and (14) if we let $Q = \|(Px)(t) - (P\bar{x})(t)\|$ we have that

$$\begin{aligned}
 Q &\leq \int_{t_0}^t \|h(s, x(s)) - h(s, \bar{x}(s))\| ds + \int_{t_0}^t \|A(x(s) - \bar{x}(s))\| \|G(s, t_0, \Phi)\| ds \\
 &\quad + \int_{t_0}^t \int_{t_0}^s \|A(\bar{x}(s))g(s, \tau, \bar{x}(\tau)) - A(x(s))g(s, \tau, x(\tau))\| d\tau ds
 \end{aligned}$$

Then

$$\begin{aligned}
 Q &\leq L_1 \int_{t_0}^t \|x(s) - \bar{x}(s)\| ds + \int_{t_0}^t \|x(s) - \bar{x}(s)\| M_2 ds \\
 &+ \int_{t_0}^t \int_{t_0}^s \|A(\bar{x}(s))g(s, \tau, \bar{x}(\tau)) - A(\bar{x}(s))g(s, \tau, x(\tau)) \\
 &+ A(\bar{x}(s))g(s, \tau, x(\tau)) - A(x(s))g(s, \tau, x(\tau))\| d\tau ds \\
 &\leq (L_1 + M_2) \int_{t_0}^t \|x(s) - \bar{x}(s)\| ds + \int_{t_0}^t \int_{t_0}^s \|A(\bar{x}(s))(g(s, \tau, \bar{x}(\tau)) - g(s, \tau, x(\tau))) \\
 &+ (A(\bar{x}(s)) - A(x(s)))g(s, \tau, x(\tau))\| d\tau ds \\
 &\leq (L_1 + M_2) \int_{t_0}^t \|x(s) - \bar{x}(s)\| ds + \int_{t_0}^t \int_{t_0}^s \|\bar{x}(s)\| L_2 \|\bar{x}(\tau) - x(\tau)\| d\tau ds \\
 &+ \int_{t_0}^t \int_{t_0}^s \|x(s) - \bar{x}(s)\| \|g(s, \tau, x(\tau))\| d\tau ds \\
 &\leq (L_1 + M_2)(t - t_0)\rho(x, \bar{x}) + \int_{t_0}^t (1 + \|\Phi(t_0)\|)\rho(x, \bar{x})(s - t_0) ds \\
 &+ \int_{t_0}^t \int_{t_0}^s \rho(x, \bar{x}) \frac{M_3}{\beta_1} d\tau ds \\
 &\leq (L_1 + M_2)\beta\rho(x, \bar{x}) + \beta(1 + \|\Phi(t_0)\|)\rho(x, \bar{x})(t - t_0) + \rho(x, \bar{x}) \frac{M_3}{\beta_1} \int_{t_0}^t (s - t_0) ds \\
 &\leq (L_1 + M_2)\beta\rho(x, \bar{x}) + \beta^2(1 + \|\Phi(t_0)\|)\rho(x, \bar{x}) + \rho(x, \bar{x}) \frac{M_3}{\beta_1} \beta^2 \\
 &\leq ((L_1 + M_2)\beta + \beta(1 + \|\Phi(t_0)\|) + \beta M_3)\rho(x, \bar{x}) \leq k\rho(x, \bar{x})
 \end{aligned}$$

which means that

$$\|(Px)(t) - (P\bar{x})(t)\| \leq k\rho(x, \bar{x})$$

for $t \in [t_0, t_0 + \beta]$ and $x, \bar{x} \in S$

So $\rho(Px, P\bar{x}) \leq \rho(x, \bar{x})$ for some $\beta > 0$ chosen in such a way that

$$k = ((L_1 + M_1) + (1 + \|\Phi(t_0)\|) + M_3)\beta < 1$$

Proving that there exist $\beta > 0$ such that Px is a contraction $[t_0, t_0 + \beta]$ and in consequence there exist a unique $x \in S$ with $Px = x$

Since the fixed points of P are the solutions of (2) the conclusion follows.

Conclusions

- (1) This theorem of existence and uniqueness illustrate a classical method to assure the existence of solutions of a differential equation, and in this case with infinite delay, but this technique can be extended to some others type of differential equations.
- (2) The existence of the solution in a local result in the interval $[t_0, t_0 + \beta]$ for some $\beta > 0$. Now using traditional methods we can extend the solution to $[t_0, +\infty)$.

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