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Positive and negative spectral projections are maps of class C^∞

Las proyecciones espectrales positivas y negativas son aplicaciones de clase C^∞

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Resumen

Sea H un espacio de Hilbert real o complejo. Denotaremos por $Gl_S(H)$ el conjunto formado por los isomorfismos auto-adjuntos limitados en H. Si $L \in Gl_S(H)$, entonces existe una descomposición $H = H_+(L) \oplus H_-(L)$, invariante por L, tal que L es positivo en $H_+(L)$ y negativo en $H_-(L)$. El objetivo principal de este trabajo es dar una prueba elemental de que $P_-, P_+ : Gl_S(H) \to L_S(H)$, $P_-(L)$, donde $P_-(L)$ y $P_+(L)$ son las proyecciones ortogonales sobre $H_-(L)$ y $H_+(L)$ respectivamente, pueden ser expresadas como

$$P_-(L) = -\frac{1}{2\pi i}\int_\Gamma (L-\lambda I)^{-1} d\lambda \quad \text{e} \quad P_+(L) = I + \frac{1}{2\pi i}\int_\Gamma (L-\lambda I)^{-1} d\lambda,$$

donde Γ es un camino cerrado diferenciable que contiene en su interior el espectro negativo de L. Usando esta representación, veremos que P_- y P_+ son aplicaciones de clase C^∞.

Palabras claves: Teoria espectral, complexificación de operadores lineales, subespacios espectrales positivos y negativos, proyección ortogonal, fórmula integral de Cauchy.

Abstract

Let H be a real or complex Hilbert space. We denote by $Gl_S(H)$ the set consisting of self-adjoint bounded isomorphism. If $L \in Gl_S(H)$, then there exist a L-invariant splitting $H = H_+(L) \oplus H_-(L)$, such that L is positive on $H_+(L)$ and negative on $H_-(L)$. The main goal of this work is to give an elementary prove of that $P_-, P_+ : Gl_S(H) \to L_S(H)$, where $P_-(L)$ and $P_+(L)$ are the orthogonal projections onto $H_-(L)$ and $H_+(L)$ respectively, can be expressed as

$$P_-(L) = -\frac{1}{2\pi i}\int_\Gamma (L-\lambda I)^{-1} d\lambda \quad \text{and} \quad P_+(L) = I + \frac{1}{2\pi i}\int_\Gamma (L-\lambda I)^{-1} d\lambda,$$

where Γ is a closed path containing the negative spectrum of L in its interior. Using this representation, we will see that P_- and P_+ are C^∞-maps.

Keywords: Spectral theory, complexification of linear operators, negative and positive, spectral subspaces, orthogonal projection, Cauchy's integral formula.

1. Introduction

Throughout this article, H shall denote a real or complex Hilbert space. Suppose that there exist a splitting $H = H_1 \oplus H_2$, where H_1 and H_2 are closed orthogonal subspaces of H. The projection onto H_1 is the operator

$$Px = x_1, \quad \text{where } x = x_1 + x_2 \in H_1 \oplus H_2.$$

Thus, $P \in L(H)$, $P^2 = P$, and, given that H_1 and H_2 are orthogonals, P is self-adjoint.

Conversely, if $P \in L(H)$, $P^2 = P$ and P is self-adjoint, then $H = \operatorname{Im} P \oplus \operatorname{Ker} P$ and $\operatorname{Im} P$ and $\operatorname{Ker} P$ are closed orthogonal subspaces of H.

Fix a $L \in L_S(H)$, where $L_S(H)$ denote the space consisting of self-adjoint bounded operators on L. Theorem 2.4 belows show that there exist a splitting

$$H = H_+(L) \oplus H_-(L) \oplus \operatorname{Ker} L$$

where $H_+(L)$ and $H_-(L)$ are L-invariants[1] closed subspaces of H, L is positive on $H_+(L)$ and negative on $H_-(L)$, i.e., $\langle Lx, x\rangle \geq 0$ for all $x \in H_+(L)$ and $\langle Lx, x\rangle \leq 0$ for all $x \in H_-(L)$. The sets $H_+(L)$ and $H_-(L)$ are called the *positive spectral subspace* and the *negative spectral subspace* of L, respectively.

Suppose that H is finite-dimensional and $L \in L(H)$ is self-adjoint. Thus $\sigma(L)$ consists of eigenvalues of L, $H_+(L)$ is spanned by eigenvectors which have positive eigenvalues and $H_-(L)$ is spanned by eigenvectors which have positive eigenvalues. Positive and negative spectral subspace in a infinite-dimensional Hilbert space are a generalization of the positive and negative eigenspaces of L.[2]

We will give an elementary proof that, for any $L \in Gl_S(H)$, the orthogonal projection onto $H_-(L)$ ($P_+(L)$), which we will be denoted by $P_-(L)$ ($P_+(L)$) can be expressed as

$$P_-(L) = -\frac{1}{2\pi i}\int_\Gamma (L - \lambda I)^{-1} d\lambda \quad \text{and} \quad P_+(L) = I + \frac{1}{2\pi i}\int_\Gamma (L - \lambda I)^{-1} d\lambda, \tag{1.1}$$

where Γ is a path containing $\sigma^-(L)$ in its interior. Moreover, using this representation, we will show that $P_-, P_+ : Gl_S(H) \to L(H)$ is a C^∞-map.

The positive and negative spectral subspaces are used for to define the relative Morse index. In fact, let $S, T \in Gl_S(H)$ such that $S - T$ is compact. One has that $H_-(S) \cap H_+(T)$ and $H_+(S) \cap H_-(T)$ have finite dimension. The *relative Morse index* of the pair (S, T) is defined by

$$\mu_{rel}(S, T) = \dim H_-(S) \cup H_+(T) - \dim H_+(S) \cap H_-(T)$$

[1] $G \subseteq H$ is L-invariant if $L(G) \subseteq G$.

[2] In the case $\dim H = \infty$, vectors in $H_+(L)$ or in $H_-(L)$ are not necessarily eigenvectors of L and points in $\sigma(L)$ are not necessarily eigenvalors of L.

(see [8], Section 2).

In [9], the differentiability of P_- is necessary for to prove there exists a *cogredient parametrix* for a family of strongly-indefinite Fredholm self-adjoint operators (see [9], Lemma 2.2). With this cogredient parametrix is defined the *spectral flow* of a path of self-adjoint Fredholm operators (see [9], Section 3).

In the next section we will remember some notions and recall several known results that will be used in the rest of the work. Furthermore, we shall prove that $Gl_S(H)$ is a Banach manifold, where $Gl_S(H)$ denote the subset of $L(H)$ consisting of self-adjoint isomorphisms.

In Section 3 we will see the Cauchy's integral formula can be generalizated for applications $L(H)$-valueds, i.e., if $L \in L(H)$ and $f : \Delta \to \mathbb{C}$ is a holomorphic application such that $\sigma(L) \subseteq \Delta$, one can define an operator $\hat{f}(L)$ by the formula

$$\hat{f}(L) = -\frac{1}{2\pi i}\int_{\Gamma} f(\lambda)(L-\lambda I)^{-1}d\lambda, \tag{1.2}$$

where $\Gamma \subseteq \Delta$ is a closed path containing $\sigma(L)$ in its interior. We shall prove that $L \mapsto \hat{f}(L)$ (whenever defined) is an application of class C^∞.

Finally, in Section 4 we will see that P_- can be represented by the formula (1.2). Consequently, P_- is of class C^∞ and, furthermore, we shall show that for any $L \in Gl_S(H)$, the derivative of P_- at L, $D(P_-)_L : L_S(H) \to L_S(H)$, is defined by

$$S \mapsto \frac{1}{2\pi i}\int_{\Gamma}(L-\lambda I)^{-1}S(L-\lambda I)^{-1}d\lambda.$$

2. Preliminaries

In the beginning of this section, we recall some notions and recall several known results of Functional Analysis that will be fundamentals to reach our main goal. Let $L(H)$ be the Banach space of bounded linear operators $L : H \to H$ with the norm

$$\|L\| = \sup_{x\in H}\frac{\|Lx\|}{\|x\|}$$

and $Gl_S(H) = \{L \in L(H) : L \text{ is a self-adjoint isomorphism}\}$.

If H is a real Banach space and $L \in L(H)$, we define the spectrum of L making use of the complexifications of H and of L (see, for example, [2], [7], [10], [4]). For any real Hilbert space H with inner product $\langle\cdot,\cdot\rangle$, we denote by $\widehat{H}$ the complexification of H, i.e., $\widehat{H} = \{x_1 + ix_2 : x_1, x_2 \in H\}$ with the operations

I. $(x_1 + ix_2) + (y_1 + iy_2) = x_1 + y_1 + i(x_2 + y_2)$ for $x_1 + ix_2, y_1 + iy_2 \in \widehat{H}$.

II. $(a + ib)(x_1 + ix_2) = ax_1 - bx_2 + i(bx_1 + ax_2)$ for $x_1 + ix_2 \in \widehat{H}$ and $a + ib \in \mathbb{C}$.

We consider the inner product

$$\langle x_1 + ix_2, y_1 + iy_2\rangle_{\widehat{H}} = \langle x_1, y_1\rangle - i\langle x_1, y_2\rangle + i\langle x_2, y_1\rangle + \langle x_2, y_2\rangle, \tag{2.1}$$

for $x_1 + ix_2, y_1 + iy_2 \in \widehat{H}$.

The *complexification* of $L \in L(H)$ is the operator $\widehat{L} \in L(\widehat{H})$, defined by

$$\widehat{L}(x+iy) = L(x) + iL(y) \text{ for all } x+iy \in \widehat{H}.$$

From definition of $\widehat{L}$, we infer that

$$\|\widehat{L}\|_{\widehat{H}} = \|L\|.$$

If H is a real Hilbert space and $L \in L(H)$, the spectrum of L is defined as the spectrum of its complexification, this is $\sigma(L) = \sigma(\widehat{L})$. It is well known that the spectrum of an operator $L \in L(H)$ is a non-empty compact subset of $\mathbb{C}$ and that

$$\|\lambda\| \leq \|L\|, \quad \text{for all } \lambda \in \sigma(L). \tag{2.2}$$

For reader's convenience, we will record some other notions that we need to obtain our principal result. Next theorem gives an important property of the spectrum of self-adjoint operator.

Theorem 2.1. *The spectrum of a self-adjoint operator $L \in L(H)$ is contained in the interval $[m, M] \subseteq \mathbb{R}$, where*

$$m = \inf_{\|x\|=1} \langle Lx, x\rangle \quad \textit{and} \quad M = \sup_{\|x\|=1} \langle Lx, x\rangle.$$

Let $L \in L(H)$ be a self-adjoint operator. We say that L is *nonnegative* if $\langle Lx, x\rangle \geq 0$, for all $x \in H$. If $\langle Lx, x\rangle > 0$ ($\langle Lx, x\rangle < 0$) for all $x \in H$, with $\|x\| = 1$, then we say that L is *positive* (*negative*). The set of positive (negative) isomorphisms in $L(H)$ is denoted by $Gl_S^+(H)$ ($Gl_S^-(H)$).

In [[5], Proposition 3.3], is proved that $Gl_S^+(H)$ is an open subset of $L_S(H)$, where $L_S(H)$ is the space consisting of self-adjoint operators. $L_S(H)$ is a real Banach space. Thus $Gl_S^+(H)$ is a real Banach manifold. It is clear that

$$Gl_S(H) = L_S(H) \cap Gl(H).$$

Then, $Gl_S(H)$ is open in $L_S(H)$ ($Gl(H)$ is open in $L(H)$). This fact show that $Gl_S(H)$ is a real Banach manifold of $L_S(H)$. Furthermore, for $L \in Gl_S(H)$, the tangent space of $Gl_S(H)$ at L is isomorphic to $L_S(H)$.

Observe that if H is a real Hilbert space and $L \in L(H)$ is nonnegative, then $\widehat{L} \in L(\widehat{H})$ is nonnegative (see (2.1)). It is not difficult to see that if L is positive, then L is injective.

Now we recall a well known result in the theory of operators in Hilbert spaces.

Lemma 2.2. *If $L \in Gl_S^+(H)$, then there exists $c > 0$ such that*

$$\inf_{\|x\|=1} \langle Lx, x\rangle \geq c.$$

Note that, if $L \in L(H)$ is a negative isomorphism, then there exists $l' < 0$ such that

$$l' \geq \sup_{\|x\|=1} \langle Lx, x\rangle.$$

In fact, by Lemma 2.2, there exists $l > 0$ such that

$$l \leq \inf_{\|x\|=1} \langle -Lx, x\rangle,$$

because $-L$ is a positive isomorphism. Since

$$\inf_{\|x\|=1} \langle -Lx, x\rangle = -\sup_{\|x\|=1} \langle Lx, x\rangle,$$

we have

$$0 > -l \geq \sup_{\|x\|=1} \langle Lx, x\rangle.$$

Proposition 2.3. *Let $L \in L(H)$ and H_1, H_2 be a closed subspaces of H, L-invariants and such that $H = H_1 \oplus H_2$. Then,*

$$\sigma(L) = \sigma(L_1) \cup \sigma(L_2),$$

where L_1 and L_2 are the restrictions of L to the subspaces H_1 and H_2, respectively.

Demostración. Let $\lambda \in \mathbb{C}$. It is not difficult to prove that $L - \lambda I$ is invertible if, and only if, the restrictions $L_1 - \lambda I|_{H_1}$ and $L_2 - \lambda I|_{H_2}$ are invertibles. Then, $\lambda \in \rho(L)$ if, only if, $\lambda \in \rho(L_1)$ and $\lambda \in \rho(L_2)$. Consequently,

$$\rho(L) = \rho(L_1) \cap \rho(L_2).$$

This fact prove that $\sigma(L) = \sigma(L_1) \cup \sigma(L_2)$. □

The following theorem (see [3], p. 336, or [4], Theorem 3.3.19.) show that there exist a splitting

$$H = H_+(L) \oplus H_-(L) \oplus \operatorname{Ker} L,$$

such that L is positive on $H_+(L)$ and negative on $H_-(L)$.

Theorem 2.4 (Spectral Decomposition)**.** *Let $L \in L(H)$ be an auto-adjoint operator. There exist a unique decomposition*

$$H = H_+(L) \oplus H_-(L) \oplus \operatorname{Ker} L$$

such that:

I. $\overline{\operatorname{Im} L} = H_+(L) \oplus H_-(L)$,

II. *$H_+(L)$ and $H_-(L)$ are L-invariants closed subspaces of H,*

III. *$H_+(L)$, $H_-(L)$ and $\operatorname{Ker} L$ are orthogonal, and*

IV. *L is positive on $H_+(L)$ and negative on $H_-(L)$.*

The subspaces $H_+(L)$ and $H_-(L)$ are called the *positive spectral subspace* and the *negative spectral subspace* of L, respectively.

Suppose that H is finite-dimensional and $L \in L(H)$ is self-adjoint. In this case, $\sigma(L)$ consists of eigenvalues of L. Moreover, $H_+(L)$ is spanned by eigenvectors which have positive eigenvalues and $H_-(L)$ is spanned by eigenvectors which positive eigenvalues. Positive and negative spectral subspace in a infinite-dimensional Hilbert space are one generalization of the eigenspaces.

We will denote by $P_-(L)$ the orthogonal projection onto $H_-(L)$. Now, we shall suppose that $L \in Gl_S(H)$. Thus $\operatorname{Ker} L = \{0\}$. Consequently, $H = H_+(L) \oplus H_-(L)$. In this case, note that the orthogonal projection onto $H_+(L)$ is $I - P_-(L)$, where I is the identity on H.

Since $H_+(L)$ and $H_-(L)$ are L-invariants, we may consider the restrictions

$$L|_{H_+(L)} : H_+(L) \to H_+(L) \quad \text{and} \quad L|_{H_-(L)} : H_-(L) \to H_-(L).$$

By Proposition 2.3, we have

$$\sigma(L) = \sigma(L|_{H_+(L)}) \cup \sigma(L|_{H_-(L)}).$$

Since $L|_{H_+(L)}$ is positive and $L|_{H_-(L)}$ is negative, by Theorem 2.1, we have

$$\sigma(L|_{H_+(L)}) = \sigma(L) \cap \mathbb{R}^+ \quad \text{and} \quad \sigma(L|_{H_-(L)}) = \sigma(L) \cap \mathbb{R}^-.$$

3. Functions of operators

The notion of integral of a complex function can be generalized to functions with values in a complex Banach space. As in the complex case, the continuous functions with values in Banach spaces are integrable. Readers who wants to learn about this notion can to read, for instance, [3], [10], [4]. In fact, suppose that $L \in L(H)$ and let $f : \Delta \to \mathbb{C}$ be a holomorphic application such that $\sigma(L) \subseteq \Delta$. Further, let $\omega \subseteq \mathbb{C}$ be an open set such that its boundary consists of a finite number of closed paths Γ_1,...,Γ_n, and

$$\sigma(L) \subseteq \omega = \bigcup_{i=1}^{n} \mathring{\Gamma}_i \subseteq \overline{\bigcup_{i=1}^{n} \mathring{\Gamma}_i} \subseteq \Delta.$$

One can define an operator $\hat{f}(L)$ by the formula

$$\hat{f}(L) = -\frac{1}{2\pi i} \int_{\partial\omega} f(\lambda)(L - \lambda I)^{-1} d\lambda. \tag{3.1}$$

The equation (3.1) is an analogous version to the Cauchy's integral formule in the context of operators in H. In Section 4, we will see that the spectral projections can be represented as the integral in (3.1).

Readers can see that the proof of following lemmas are not difficult. These are some properties of the above integral and they can be proved as in the complex-valued applications case.

Lemma 3.1. *Let $f : \Delta \to H$ be a holomorphic application. Then for any rectifiable path $\Gamma \subseteq \Delta$,*

$$\left\| \int_\Gamma f(\lambda) d\lambda \right\| \leq Ml,$$

where $M = \sup_{\lambda \in \Gamma} |f(\lambda)|$ and l is the length of Γ.

Lemma 3.2. *Let L be an operator in $L(H)$. Suppose that $f : \Delta \to \mathbb{C}$ and $g : \Delta \to \mathbb{C}$ are holomorphic applications and $\sigma(L) \subseteq \Delta$. If $h : \Delta \to \mathbb{C}$ is defined by $h(\lambda) = f(\lambda)g(\lambda)$ (product of complex applications), then*

$$\hat{h}(L) = \hat{f}(L)\hat{g}(L) := \hat{f}(L)\hat{g}(L).$$

Demostración. See [10], Lemma 6.15. □

For any $L \in L(H)$, the application $R : \rho(L) \to L(E)$, defined by $R(\lambda) = (L-\lambda I)^{-1}$ for $\lambda \in \rho(L)$, is holomorphic (see [3], Chapter I, §5). Consequently, for any path $\Gamma \subseteq \rho(L)$ containing $\sigma(L)$ is its interior, we have

$$\lambda \mapsto \|(L-\lambda I)^{-1}\|, \quad \text{for } \lambda \in \Gamma,$$

is continuous. Since Γ is compact, we have

$$0 < \min_{\lambda\in\Gamma} \|(L-\lambda I)^{-1}\| \leq \max_{\lambda\in\Gamma} \|(L-\lambda I)^{-1}\| = M_1 < \infty.$$

Then, there exists $M > 0$ such that if T is sufficiently close to L,

$$\|(T-\lambda I)^{-1}\| \leq M. \tag{3.2}$$

We give an elementar proof of the application $L \to \hat{f}(L)$ (whenever defined) is a C^1 map.

Theorem 3.3. *Let O an open set in $L(H)$ and $f : \Delta \to \mathbb{C}$ an holomorphic application with $\sigma(L) \subseteq \Delta$ for any $L \in O$. Suppose there exist a path $\Gamma \subseteq \Delta$ (or finitely many paths $\Gamma_1, \Gamma_2, ..., \Gamma_n$) containing $\sigma(L)$ in its interior, for any $L \in O$. Then $\hat{f} : O \to L(H)$ is a map of class C^1. Furthermore, for each $L \in O$, $D\hat{f}_L : L_S(H) \to L_S(H)$ is defined by*

$$S \mapsto \frac{1}{2\pi i}\int_\Gamma f(\lambda)(L-\lambda I)^{-1}S(L-\lambda I)^{-1}d\lambda.$$

Demostración. Let $L \in O$ and $S \in L(H)$. Then,

$$\begin{aligned}
\lim_{t\to 0}\frac{\hat{f}(L+tS)-\hat{f}(L)}{t} &= \lim_{t\to 0}\frac{1}{2\pi i}\int_\Gamma f(\lambda)\frac{(L+tS-\lambda I)^{-1}tS(L-\lambda I)^{-1}}{t}d\lambda \\
&= \lim_{t\to 0}\frac{1}{2\pi i}\int_\Gamma f(\lambda)(L+tS-\lambda I)^{-1}S(L-\lambda I)^{-1}d\lambda \\
&= \frac{1}{2\pi i}\int_\Gamma f(\lambda)(L-\lambda I)^{-1}S(L-\lambda I)^{-1}d\lambda
\end{aligned}$$

(the continuity of $\lambda \mapsto f(\lambda)(L-\lambda I)^{-1}S(L-\lambda I)^{-1}$ implies the existence of this integral). Consequently, $D\hat{f}_L(S) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(L-\lambda I)^{-1}S(L-\lambda I)^{-1}d\lambda$. It is not difficult to prove that $D\hat{f}_L$ is linear and, by Lemma 3.1, $D\hat{f}_L$ is bounded. Thus $D\hat{f}_L \in L(L(H))$.

Finally, we prove that $D\hat{f}_L$ depends continuously on L. Let $\delta > 0$ such that if $T \in O$ with $\|T-L\| < k$, $\|(T-\lambda I)^{-1}\| \leq M$ (see Equation (3.2)). Then, for any $S \in L_S(H)$, $D\hat{f}_T(S) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(T-\lambda I)^{-1}S(T-\lambda I)^{-1}d\lambda$ and thus,

$$[D\hat{f}_L - D\hat{f}_T](S) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)[(L-\lambda I)^{-1}S(L-\lambda I)^{-1} - (T-\lambda I)^{-1}S(T-\lambda I)^{-1}]d\lambda.$$

In [6] we prove that

$$\|(L-\lambda I)^{-1}S(L-\lambda I)^{-1} - (T-\lambda I)^{-1}S(T-\lambda I)^{-1}\| \leq 2M^3\|T-L\|\|S\|,$$

for any $S \in L_S(H)$ and $\lambda \in \Gamma$. Then, it follows from Lemma 3.1 that

$$\|[D\hat{f}_L - D\hat{f}_T](S)\| \leq \frac{lM^3\tilde{\lambda}}{\pi}\|T-L\|\|S\|,$$

where $\tilde{\lambda} = \text{máx}\{|f(\lambda)| : \lambda \in \Gamma\}$. This fact implies that $\|D\hat{f}_L - D\hat{f}_T\| \leq \frac{lM^3\tilde{\lambda}}{\pi}\|T - L\|$ and consequently, $D\hat{f}_L$ depends continuously on L. This finishes the proof of the theorem. $\square$

We shall denote by $B(L, \delta)$ the open ball $\{T \in L(H) : \|L - T\| < \delta\}$. For each $\lambda \in \Gamma$, let

$$\begin{aligned} \phi_\lambda : B(L, \delta) &\to L(H) \\ R &\mapsto (R - \lambda I)^{-1}. \end{aligned}$$

It follows from [[1], Lemma 1.3.15] that ϕ_λ is of class C^∞ and, moreover, for any $\lambda \in \Gamma$ fixed and $R \in B(L, k)$, $D(\phi_\lambda)_R(S) = -(R - \lambda I)^{-1}S(R - \lambda I)^{-1}$, for all $S \in L(H)$. We denote by $D^{(n)}(\phi_\lambda)_R$ the n-th derivative of ϕ_λ at R. It follows from Theorem 3.3 that

$$D\hat{f}_L(S) = -\frac{1}{2\pi i}\int_\Gamma f(\lambda)D(\phi_\lambda)_L(S)d\lambda \quad \text{for every } S \in L_S(H).$$

Using induction on n, we can prove that $D^{(n)}\hat{f}_L(S)$ is defined by

$$S \mapsto -\frac{1}{2\pi i}\int_\Gamma f(\lambda)D^{(n)}(\phi_\lambda)_L(S)d\lambda \quad \text{for every } S \in L_S(H).$$

Consequently, $\hat{f}$ is a a map C^∞.

4. Differentiability and Integral Representation of P_- and P_+

In this section, we will see that for any $L \in Gl_S(H)$, there exists a path $\Gamma \subseteq \{x + iy \in \mathbb{C} : x < 0\}$ containing the negative spectrum of L in its interior, and moreover, we can define

$$P(L) = -\frac{1}{2\pi i}\int_\Gamma (L - \lambda I)^{-1}d\lambda.$$

The autor of the paper under review provides an alternative and elementary proof that $P(L)$ is the orthogonal projection onto $H_-(L)$, the negative spectral subspace of L (in the complex case). Furthermore, if T is sufficiently close to L, then, for the same Γ, $P(T) = -\frac{1}{2\pi i}\int_\Gamma (T - \lambda I)^{-1}d\lambda$ is well defined. Thus, by Theorem 3.3, we conclude that the negative spectral projection, when restricted to $Gl_S(H)$, is an application of class C^1.

At the end of this section, using the complexification defined in the previous section, we will discuss the case when H is real.

First we assume that H is a complex Hilbert space. For each $L \in Gl_S(H)$, there exists $c > 0$ such that $\|Lx\| \geq c\|x\|$ for any $x \in H$. For the rest of this work, we consider $L \in Gl_S(H)$ fixed and Γ will be the circumference (positively oriented) with center $a = -\frac{c/2+\|L\|+1}{2}$ and radius $r = \frac{c/2+\|L\|+1}{2}$. We prove the next lemma.

Lemma 4.1. Γ *contains in its interior the negative espectrum of* T, *for every* $T \in B(L, c/2)$,

Demostración. Let $T \in B(L, c/2)$. We show that, if $\lambda \in \mathbb{R}$ with $|\lambda| < c/2$, then $\lambda \in \rho(T)$ (consequently, 0 is not an accumulation point of the set $\bigcup_{T\in B(L,c/2)} \sigma(T)$). Choose $\lambda \in \mathbb{R}$ with $|\lambda| < c/2$ and $x \in H$ with $\|x\| = 1$. Note that

$$\|Tx\| \geq \|Lx\| - \|Lx - Tx\| \geq c - c/2,$$

i.e., $\|Tx\| \geq c/2$. Thus,

$$\|(T-\lambda I)x\| \geq \|Tx\| - \|\lambda x\| \geq \frac{c}{2} - |\lambda| > 0.$$

This fact prove that $\mathrm{Ker}(T-\lambda I) = \{0\}$. Then, $\mathrm{Im}(T-\lambda I)$ is closed (See [2], Corollary 2.25). Furthermore, since $T-\lambda I$ is self-adjoint, we have $\mathrm{Im}(T-\lambda I) = [\mathrm{Ker}(T-\lambda I)^*]^\perp = [\mathrm{Ker}(T-\lambda I)]^\perp = H$. Consequently, $\lambda \in \rho(T)$.

On the other hand, for any $x \in H$, with $\|x\| = 1$, we have

$$\langle Tx, x\rangle = \langle (T-L)x, x\rangle + \langle Lx, x\rangle < c/2 + \|L\|.$$

Then, $\|T\| < c/2 + \|L\|$. It follows from (2.2) that

$$\sigma(T) \subseteq [-(c/2 + \|L\|), c/2 + \|L\|] \quad \text{for } T \in B(L, c/2).$$

This finish the proof. □

If follows from Lemma 4.1 that, for any $T \in B(L, c/2)$,

$$P(T) = -\frac{1}{2\pi i}\int_\Gamma (T-\lambda I)^{-1} d\lambda$$

is well defined. Let $\Delta_1 = \{x+iy \in \mathbb{C} : x < 0\}$ and $\Delta_2 = \{x+iy \in \mathbb{C} : x > 0\}$. The application $\gamma : \Delta_1 \cup \Delta_2 \to \mathbb{C}$ defined as $\gamma|_{\Delta_1} \equiv 1$ and $\gamma|_{\Delta_2} \equiv 0$ is holomorphic in $\Delta_1 \cup \Delta_2$. Note that $P(T) = \hat{\gamma}(T)$. Then, for Theorem 3.3, $P : Gl_S(H) \to L(H)$ is a C^∞ map.

As we saw in Section 2, the tangent space of $Gl_S(H)$ at L is isomorphic to $L_S(H)$. By abuse of notation we will consider that $T_L(Gl_S(H)) = L_S(H)$, where $T_L(Gl_S(H))$ denotes the tangent space of $Gl_S(H)$ at L. Consequently, the derivative of P at L is the operator $DP_L : L_S(H) \to L(H)$ defined as

$$S \mapsto \frac{1}{2\pi i}\int_\Gamma (L-\lambda I)^{-1} S (L-\lambda I)^{-1} d\lambda.$$

Now, we prove that $P(L)$ is the orhogonal projection onto $H_-(L)$. In fact, let $\sigma(L) = \sigma^+ \cup \sigma^-$ be the spectrum of L, where $\sigma^+ \subset \mathbb{R}^+$ and $\sigma^- \subset \mathbb{R}^-$. By Lemma 3.2 we have $P(L)^2 = P(L)$ ($\gamma^2 = \gamma$). Hence,

$$H = \mathrm{Im}\, P(L) \oplus \mathrm{Ker}\, P(L)$$

and, furthermore, $\mathrm{Im}\, P(L)$ and $\mathrm{Ker}\, P(L)$ are L-invariants. Moreover, considering the restrictions

$$L|_{\mathrm{Im}\, P(L)} : \mathrm{Im}\, P(L) \to \mathrm{Im}\, P(L)$$

and

$$L|_{\mathrm{Ker}\, P(L)} : \mathrm{Ker}\, P(L) \to \mathrm{Ker}\, P(L),$$

we have:

Lemma 4.2. $\sigma(L|_{\mathrm{Im}\, P(L)}) = \sigma^-$ *and* $\sigma(L|_{\mathrm{Ker}\, P(L)}) = \sigma^+$.

Demostración. See [10], p. 141, Theorem 6.19. □

To prove that $P(L)$ is the orthogonal projection onto $H_-(L)$, we have to show that $P(L)$ is self-adjoint and $\operatorname{Im} P(L) = H_-(L)$. In fact, for any self-adjoint operator S and f a holomorphic application, such that $\hat{f}(S)$ is well defined, we have $\hat{f}(S)$ is self-adjoint[3]. Consequently, $P(L)$ is self-adjoint. It follows that $\operatorname{Im} P(L)$ and $\operatorname{Ker} P(L)$ are orthogonals.

On the other hand, by [[5], Lemma 5.3], one have that, if S is self-adjoint, then

$$\min_{\lambda\in\sigma(S)} \lambda = \inf_{\|x\|=1} \langle Sx, x\rangle.$$

Thus,

$$L|_{\operatorname{Ker} P(L)} \text{ is positive} \tag{4.1}$$

(see Lemma 4.2). Similarly we can prove that

$$\max_{\lambda\in\sigma(L|_{\operatorname{Im} P(L)})} \lambda = \sup_{\|x\|=1} \langle L|_{\operatorname{Im} P(L)} x, x\rangle,$$

and therefore

$$L|_{\operatorname{Im} P(L)} \text{ is negative.} \tag{4.2}$$

It follows from (4.1), (4.2), and the uniqueness in Theorem 2.4 that:

Lemma 4.3. $\operatorname{Im} P(L) = H_-(L)$.

Consequently, $P(L)$ is the orthogonal projection onto $H_-(L)$.

The integral in (3.1) is only valid when H is complex. Using the complexification of an operator in a real Hilbert space, we can represent $P(L)$ in a similar integral. We will devote the rest of this section to this construction. In fact, suppose that H is a real Hilbert space and $L \in Gl_S(H)$. Let $\widehat{L} \in L(\widehat{H})$ be the complexification of L. It is not difficult to prove that $\widehat{L}$ is a self-adjoint isomorphism (see (2.1)). Thus,

$$\widehat{H} = H_+(\widehat{L}) \oplus H_-(\widehat{L}).$$

If $\widehat{x} = x + iy \in \widehat{H}$, where $x, y \in H$, the *real part* of $\widehat{x}$ is defined by $Re\widehat{x} = x$. It is not immediate see that $Re(P(\widehat{L}))$ is the orthogonal projection onto $H_-(L)$. We will prove this fact.

For any subset Ω of $\widehat{H}$, we define

$$Re(\Omega) = \{Re\widehat{x} : \widehat{x} \in \Omega\}.$$

It is clear that $\widehat{L}|_{Re(\Omega)} = L|_{Re(\Omega)}$. We will identify $L : H \to H$ with the restriction $\widehat{L}|_{Re(\widehat{H})} : Re(\widehat{H}) \to Re(\widehat{H})$.

Note that $\widehat{L}(\overline{H_+(\widehat{L})}) \subseteq \overline{H_+(\widehat{L})}$, where

$$\overline{H_+(\widehat{L})} = \{x_1 - ix_2 : x_1 + ix_2 \in H_+(\widehat{L}), x_1, x_2 \in H\}.$$

In fact, if $\widehat{x} = x_1 + ix_2 \in H_+(\widehat{L})$, with $x_1 \in H$ and $x_2 \in H$, then

$$\widehat{L}\overline{\widehat{x}} = \widehat{L}(x_1 - ix_2) = Lx_1 - iLx_2 = \overline{Lx_1 + iLx_2} = \overline{\widehat{L}\widehat{x}}.$$

[3]We prove in [[5], Theorem 5.4] that $\hat{\gamma}(L)$ is self-adjoint. Analoguosly, we can prove that $\hat{f}(S)$ is self-adjoint whenever S is self-adjoint, f is holomorphic and $\hat{f}(S)$ is well defined.

Since $\widehat{L}\widehat{x} \in H_+(\widehat{L})$, $\overline{\widehat{L}\widehat{x}} \in \overline{H_+(\widehat{L})}$.

Now, we shall prove that $\widehat{L}$ is positive on $\overline{H_+(\widehat{L})}$. Let $\overline{\widehat{x}} \in \overline{H_+(\widehat{L})}$, whit $\widehat{x} \in H_+(\widehat{L})$ and $\widehat{x} \neq 0$. It follows from (2.1) that $\langle \overline{\widehat{L}\widehat{x}}, \overline{\widehat{x}} \rangle = \overline{\langle \widehat{L}\widehat{x}, \widehat{x} \rangle}$. Thus, since $\langle \widehat{L}\widehat{x}, \widehat{x} \rangle > 0$,

$$\langle \widehat{L}\overline{\widehat{x}}, \overline{\widehat{x}} \rangle = \langle \overline{\widehat{L}\widehat{x}}, \overline{\widehat{x}} \rangle = \overline{\langle \widehat{L}\widehat{x}, \widehat{x} \rangle} > 0.$$

Similarly, $\widehat{L}$ is negative on $\overline{H_-(\widehat{L})}$. Consequently, by the uniqueness in Theorem 2.4, we have

$$\overline{H_+(\widehat{L})} = H_+(\widehat{L}) \quad \text{and} \quad \overline{H_-(\widehat{L})} = H_-(\widehat{L}). \tag{4.3}$$

Proposition 4.4. *$H = Re(H_+(\widehat{L})) \oplus Re(H_-(\widehat{L}))$, and, moreover,*

$$H_+(L) = Re(H_+(\widehat{L})) \quad \textit{and} \quad H_-(L) = Re(H_-(\widehat{L})).$$

Demostración. For $x \in H$, $\widehat{x} = x + i0 \in \widehat{H}$. Therefore, there exists unique $\widehat{x}_+ \in H_+(\widehat{L})$ and $\widehat{x}_- \in H_-(\widehat{L})$ such that

$$\widehat{x} = \widehat{x}_+ + \widehat{x}_-. \tag{4.4}$$

This implies that

$$\widehat{x} = Re\widehat{x} = Re\widehat{x}_+ + Re\widehat{x}_-. \tag{4.5}$$

By (4.3) we have $Re\widehat{x}_+ = (\widehat{x}_+ + \overline{\widehat{x}_+})/2 \in H_+(\widehat{L})$ and $Re\widehat{x}_- = (\widehat{x}_- + \overline{\widehat{x}_-})/2 \in H_-(\widehat{L})$. It follows from (4.5) and the uniqueness in (4.4) that $Re\widehat{x}_+ = \widehat{x}_+$, $Re\widehat{x}_- = \widehat{x}_-$. This fact prove that

$$H = Re(H_+(\widehat{L})) + Re(H_-(\widehat{L})).$$

Furthermore, for any $x \in Re(H_+(\widehat{L}))$,

$$Lx = \widehat{L}Rex = Re\widehat{L}x \in Re(H_+(\widehat{L})).$$

Thus, $Re(H_+(\widehat{L}))$ is L-invariant.

Note that

$$Re(H_+(\widehat{L})) = \{(\widehat{x} + \overline{\widehat{x}})/2 : \widehat{x} \in H_+(\widehat{L})\} \subseteq H_+(\widehat{L}) + \overline{H_+(\widehat{L})} = H_+(\widehat{L}).$$

Consequently, L is positive on $Re(H_+(\widehat{L}))$.

Similarly we can to prove that $Re(H_-(\widehat{L})) \subseteq H_-(\widehat{L})$ is L-invariant and L is negative on $Re(H_-(\widehat{L}))$.

The above facts show that $H = Re(H_+(\widehat{L})) \oplus Re(H_-(\widehat{L}))$, and, moreover, $H_+(L) = Re(H_+(\widehat{L}))$ and $H_-(L) = Re(H_-(\widehat{L}))$. □

In conclusion, we have:

Theorem 4.5. *If H is a real Hilbert space and $L \in Gl_S(H)$, the orthogonal projection onto $H_-(L)$ is given by*

$$Re(P(\widehat{L})) = Re\left(-\frac{1}{2\pi i}\int_\Gamma (\widehat{L} - \lambda I)^{-1} d\lambda\right).$$

Demostración. It follows from Lemmas 4.1-4.2 and Proposition 4.4. □

Note that $P_+(L)$, the projection onto $H_+(L)$, is $I - P(L)$. Then P_+ is a map of class C^∞ and, moreover, for any $L \in Gl_S(H)$, $D(P_+)_L : L_S(H) \to L_S(H)$ is defined by

$$S \mapsto -\frac{1}{2\pi i}\int_\Gamma (L-\lambda I)^{-1} S (L-\lambda I)^{-1} d\lambda.$$

When H is real, we have

$$P_+(L) = I - Re(P(\widehat{L})) = I + Re\left(\frac{1}{2\pi i}\int_\Gamma (\widehat{L}-\lambda I)^{-1} d\lambda\right).$$

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