Basic and Classic properties in the $\mathcal{B}_F$-spaces

Propiedades Básicas y Clásicas en los $\mathcal{B}_F$-espacios

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Abstract

$\mathcal{B}_F$-spaces determine a class between the class of pseudocompact spaces and the class of $k_R$-pseudocompact spaces. We present an alternative proof of the theorem 3.5 enunciated in [3] and describe their main properties.

Keywords: $k_R$-space, $\mathcal{B}_F$-spaces, pseudocompact spaces.

1. Introduction

The class of $\mathcal{B}_F$-spaces lies between the class of pseudocompact spaces and the class of pseudocompact $k_R$-spaces. The definition of $\mathcal{B}_F$-spaces 2.2 was introduced by Frolik in [(3), 3.5.1], where he proves that
such spaces are productively pseudocompact. The class was later studied by Noble [7], who doesn’t give it
a name but denotes it by \( \mathfrak{B}^* \) (\( \mathfrak{B} \) is used for the class of productively pseudocompact spaces by both Frolík
and Noble.)

It has several attractive properties like the following:

a) \( \mathfrak{B}_F \)-spaces are productively pseudocompact;

b) \( \mathfrak{B}_F \)-spaces are closed under finite products;

c) Every product of \( \mathfrak{B}_F \)-spaces is pseudocompact;

d) \( \mathfrak{B}_F \)-spaces are closed under continuous images;

e) Every space containing a dense \( \mathfrak{B}_F \)-subspace is itself \( \mathfrak{B}_F \)-spaces

We think all of these facts prove that this is a challenging area in point set topology.

2. Preliminary

The terminology of R. Engelking [2] and J. Kelley [6], General Topology, is used throughout.
All spaces consider in this paper are Tychonoff, i.e., completely regular and Hausdorff.

**Definition 2.1.** A space \( X \) is said to be :

i) pseudocompact (see Hewitt [4]) if (and only if) every real continuous function on \( X \) is bounded,
or equivalently, if every real continuous bounded function assumes its bounds. A completely regular
space \( X \) is pseudocompact if and only if every locally finite family of its open subsets is finite, or
equivalently, if there exists no locally finite sequence of its non-void open subsets.

ii) \( k_R \)-space(see Noble [7]) when every real-valued function with domain \( X \) is continuous if its restriction
to each compact subset of \( X \) is continuous.

Recall that a space \( X \) is called a \( k \)-space provided each subset of \( X \) which meets every compact set in
a relatively closed set is itself closed, and that associated with each space \( X \) there is a unique \( k \)-space \( kX \)
having the same underlying set and the same compact sets as \( X \) (see [7]).

The following definition is based on Frolík’s condition [[3], 3.5.1] which turns out to be equivalent. \( \mathfrak{B}_F \)-spaces.

**Definition 2.2.** A space \( X \) is a \( \mathfrak{B}_F \)-space if for every sequence \( U_1, U_2, \ldots \) of non-empty open sets, there
exists a compact set \( K \subseteq X \) such that \( K \cap U_n \neq \emptyset \) for infinitely many indices \( n \).

We obtain an equivalent definition if we suppose that the open sets \( U_n \) are mutually disjoint. To prove
this fact, we need a Lemma.

**Lemma 2.3.** (see also [7]) Let \( U_1, U_2, \ldots \) be a point finite sequence of non-empty open sets in a space \( X \).
Then there exists a sequence \( T_1, T_2, \ldots \) of mutually disjoint non-empty open sets in \( X \) and an increasing
sequence \( n_0 = 0 < n_1 < n_2 \cdots \) such that \( T_i \subseteq \bigcup_{j=n_{i-1}+1}^{n_i} U_j \) for each \( i \in \omega \)

\(^1\) (The space \( kX \) is formed by adjoining to the topology on \( X \) all those subsets whose complements meet each compact set in a
relatively closed set.) When \( X \) is a \( T_1 \)-space, \( kX \) is also a \( T_1 \)-space; in fact, the identity map from \( kX \) to \( X \) is always continuous.
We prove now the equivalence of the two definitions. Every Lemma 2.7. Proposition 2.6.

subset of set $U$. We may suppose that the sequence Definition 2.5.

For each $i \in \omega$, a point $x_i \in U_{n_i} \cup \bigcup_{j \neq i} U_j$. Choose an open set $W_1$ such that $x_1 \in W_1 \subseteq U_{n_1}$. Since $x_2 \notin W_1$, there exists an open set $W_2$ such that $x_2 \in W_2 \subseteq U_{n_2} \setminus W_1$. Now, since $x_3 \notin W_1 \cup W_2$, there exists an open set $W_3$ such that $x_3 \in W_3 \subseteq U_{n_3} \setminus (W_1 \cup W_2)$. Continuing this process, we may construct a sequence $W_1, W_2, \ldots$ of mutually disjoint non-empty open sets such that $W_i \subseteq U_{n_i}$ for each $i \in \omega$ and we are thru in this case. Suppose then that no subsequence of $U_1, U_2, \ldots$ is irreducible. Therefore, we may find integers $n_0 = 0 < n_1 < n_2 < \cdots$ such that if $W_i = \bigcup\{U_j: n_{i-1} < j \leq n_i\}$, then $W_1 \supsetneq W_2 \supsetneq W_3 \supsetneq \cdots$. If a subsequence of the $W_i$'s is made of clopen sets, say $W_{k_1}, W_{k_2}, \ldots$ the sequence $\{W_{k_i} - W_{k_{i-1}}: i = 1, 2, \ldots\}$ satisfies our requirements. If only finitely many of the $W_i$’s are clopen, we may remove them and suppose, with no loss of generality, that $W_i \neq W_{i-1}$ for each $i \in \omega$. If for some strictly increasing sequence $0 < n_1 < n_2 < \cdots$ we have $W_{n_i} - W_{n_{i-1}} \neq \emptyset$ for each $i \in \omega$, we define $T_i = W_{n_i} - W_{n_{i-1}}$ and the sequence of open sets $T_1, T_2, \ldots$ satisfies our requirements. If for only finitely many indices $i \in \omega$, we have $W_i - W_{i-1} \neq \emptyset$, we may remove the corresponding $W_i$ and suppose then that $W_{i+1}$ is dense in $W_i$ for each $i \in \omega$. Take a point $x_1 \in W_1 \setminus W_2$ and let $T_1$ be an open set such that $x_1 \in T_1 \subseteq T_2 \subseteq W_2$. The set $T_1 \cap W_2$ is then open and infinite. Select two different points $x_2, p_2 \in T_1 \cap W_2$ and let $T_2$ be an open set such that $x_2 \in T_2 \subseteq T_2' \subseteq T_1 \cap (W_2 - \{p_2\})$. Take now two different points $x_3, p_3 \in T_2 \cap W_3$ and let $T_3$ be an open set such that $x_3 \in T_3 \subseteq T_3' \subseteq T_2 \cap (W_3 - \{p_3\})$. It is clear now how to continue this process indefinitely. The required sequence is now $(T_i - T_{i-1}: i \in \omega)$. We prove now the equivalence of the two definitions.

Proposition 2.4. In an arbitrary space $X$, the following two conditions are equivalent:

1) $X$ is a $\mathcal{S}_F$-space.

2) For every open sequence $W_1, W_2, \ldots$ of mutually disjoint non-empty open subsets of $X$, there exists a compact set $L \subseteq X$ such that $L \cap W_n \neq \emptyset$ for infinitely many indices $n$.

Proof. We just have to prove that 2) $\Rightarrow$ 1). Let $U_1, U_2, \ldots$ be a sequence on non-empty open sets of $X$. We may suppose that the sequence $U_1, U_2, \ldots$ is point finite, because otherwise we could take the compact set $K$ as a singleton. By (2.3), there exists a sequence $T_1, T_2, \ldots$ of mutually disjoint non-empty open sets in $X$ and a strictly increasing sequence $n_0 = 0 < n_1 < n_2 < \cdots$ such that $T_i \subseteq \bigcup_{j=n_{i-1}+1}^{n_i} U_j$ for each $i \in \omega$. By property 2), there exists a compact set $K \subseteq X$ such that $K \cap T_i \neq \emptyset$ for infinitely many indices $i \in \omega$. Hence, $K \cap U_j \neq \emptyset$ for infinitely many indices $j \in \omega$ and the proof is complete.

Definition 2.5. A subset $A$ of a space $X$ is C-discrete (respect to $X$) if for each $x \in A$ we may find an open set $U_x$ containing $x$ and such that the family $\{U_x: x \in A\}$ is discrete (respect to $X$).

A well known characterization of pseudocompactness is the following:

Proposition 2.6. [see [3]] A space $X$ is pseudocompact if and only if every C-discrete subset of $X$ is finite.

(2.6) implies immediately:

Lemma 2.7. Every $\mathcal{S}_F$-space $X$ is pseudocompact.

Proof. Suppose, on the contrary, there exists an infinite discrete sequence $U_1, U_2, \ldots$ of non-empty open subset of $X$. Let $K \subseteq X$ be a compact set such that $K \cap U_n \neq \emptyset$ for every $n \in L$, where $L \subseteq \omega$ and $|L| = \omega$. For each $n \in L$, select a point $x_n \in K \cap U_n$. Then the set $A = \{x_n: n \in L\}$ is an infinite C-discrete subset of $K$, contradicting (2.6).
We call point \( x \) in \( X \) is a \( k \)-point if each open subset of \( kX \) which contains \( x \) is a neighborhood of \( x \). Clearly \( X \) is a \( k \)-space if and only if each point in \( X \) is a \( k \)-point. Recall that a point \( x \) in \( X \) is called a \( P \)-point if each \( G_\delta \) containing \( x \) is a neighborhood of \( x \). We call point \( x \) in \( X \) is a \( k_R \)-point if each real-valued function on \( X \) which is continuous on compact sets is continuous at \( x \) (see [7]).

We have the following result.

**Proposition 2.8.** (see [7], theorem 2.2) If \( X \) is pseudocompact and each point of \( X \) is either a \( P \)-point or a \( k_R \)-point, then \( X \) is a \( \mathcal{B}_F \)-space.

**Proof.** Suppose \( X \) is not \( \mathcal{B}_F \)-space, let \( \{U_n\} \) be a countable collection of disjoint open sets only finitely many of which meet any single compact set and construct and bounded function \( f \) (see [[7], theorem 2.1]). Since \( f \) is continuous on compact sets it is continuous at each \( k_R \)-point of \( X \), and \( f \) is continuous at \( P \)-point in \( X \) \( \setminus \bigcup_n U_n \) since it is zero on a neighborhood of such that a point. Finally, since \( f|U_n = f_n \), \( f \) is continuous on \( \bigcup_n U_n \) and therefore \( f \) is continuous. Since \( X \) is pseudocompact, this is a contradiction so \( X \) is a \( \mathcal{B}_F \)-space.

The following result is not a new result (Noble uses this fact in the proof of [[7], Theorem 2.1]) but the author explicitly formulated and proved.

**Proposition 2.9.** Let \( \varphi : X \rightarrow Y \) be a continuous map of the \( \mathcal{B}_F \)-space \( X \) onto the space \( Y \). Then \( Y \) is a \( \mathcal{B}_F \)-space.

**Proof.** Let \( V_1, V_2, \ldots \) be a sequence of non-empty open sets in \( Y \). For each \( n \in \omega \), define \( U_n = \varphi^{-1}(V_n) \). By the continuity of \( \varphi \), each \( U_n \) is open in \( X \). Since \( X \) is a \( \mathcal{B}_F \)-space, there exists a compact set \( L \subseteq X \) such that \( L \cap U_n \neq \emptyset \) for infinitely many indices \( n \in \omega \). Therefore, \( \varphi(L) \) is compact and \( \varphi(L) \cap V_n \neq \emptyset \) for infinitely many indices \( n \in \omega \), i.e. \( Y \) is a \( \mathcal{B}_F \)-spaces.

We have also the following result:

**Theorem 2.10.** If a space \( X \) has a dense subspace \( Y \) which is a \( \mathcal{B}_F \)-space, then \( X \) itself is a \( \mathcal{B}_F \)-space.

**Proof.** Let \( V_1, V_2, \ldots \) be a sequence of non-empty open sets in \( X \). For each \( n \in \omega \), define \( U_n = Y \cap V_n \). Then \( U_n \) is an open non-empty subset of \( Y \). By hypothesis, there exists a compact set \( K \subseteq Y \) such that \( K \cap U_n \neq \emptyset \) for infinitely many indices \( n \in \omega \). Hence, \( K \cap V_n \neq \emptyset \) for infinitely many indices and the proof is complete.

We finish this preliminary section proving the following result:

**Theorem 2.11.** Every finite product of \( \mathcal{B}_F \)-spaces is a \( \mathcal{B}_F \)-space.

**Proof.** It is enough to prove that if \( X, Y \) are \( \mathcal{B}_F \)-spaces, then \( X \times Y \) is also \( \mathcal{B}_F \)-space. Let \( W_s = U_s \times V_s \) be non-empty basic open sets in \( X \times Y \). Let \( K_1 \subseteq X \) be a compact set in \( X \) such that \( K_1 \cap U_s \neq \emptyset \) for every \( s \in L_1 \), where \( L_1 \subseteq \omega \) and \( |L_1| = \omega \). Let now \( K_2 \subseteq Y \) be a compact set in \( Y \) such that \( K_2 \cap V_s \neq \emptyset \) for every \( s \in L_2 \), with \( L_2 \subseteq L_1 \), \( |L_2| = \omega \). Therefore \( K = K_1 \times K_2 \) is compact and satisfies \( K \cap W_s \neq \emptyset \) for every \( s \in L_2 \). The proof is then complete.

3. Main results.

In this section we prove the two properties of \( \mathcal{B}_F \)-spaces mentioned in the introduction which were not proved in the last section.

In the following result we present an alternative proof of the theorem 3.5 enunciated in [3].
Theorem 3.1. Let be a \( \mathcal{B}_F \)-space and let \( Y \) be pseudocompact. Then \( X \times Y \) is pseudocompact.

Proof. Suppose, on the contrary, that \( X \times Y \) is not pseudocompact. By (2.6), there exists an infinite discrete family \( U_1, U_2, \ldots \) of non-empty open sets in \( X \times Y \). Let \( \pi: X \times Y \to X \) be the projection onto the first factor. There exists an index \( n_1 \in \omega, n_1 \geq 2 \), such that \( \pi(U_1) \cap \pi(U_2) \cap \cdots \cap \pi(U_{n_1}) = \emptyset \); otherwise, there would exist a point \( z \in \bigcap_{n=1}^{\infty} \pi(U_n) \) and the set \( \{z\} \times Y \) would be a pseudocompact subset of \( X \times Y \) which would intersect every \( U_n \), a fact which, by (2.6), cannot occur. Pick a minimum \( n_1 \in \omega \). Therefore, \( \bigcap_{n=1}^{n_1-1} \pi(U_n) \neq \emptyset \). Reasoning in a similar way, we may find a minimum integer \( n_2 \geq n_1 + 2 \) such that \( \pi(U_{n_1+1}) \cap \cdots \cap \pi(U_{n_2}) = \emptyset \) and continue this process indefinitely. For each \( k \in \omega \), \( W_k = \pi(U_{n_1+1}) \cap \cdots \cap \pi(U_{n_k}) \) is a non-empty open subset of \( X \). Since \( X \) is a \( \mathcal{B}_F \)-space, there exists a compact set \( K \subseteq X \) such that \( K \cap W_k \neq \emptyset \) for infinitely many indices \( k \). But then \( K \times Y \) is a pseudocompact subset of \( X \times Y \) which intersects \( U_n \) for infinitely many indices \( n \in \omega \), and this is a contradiction.

Theorem 3.2. [See [7], Theorem 3.4] Every topological product of \( \mathcal{B}_F \)-spaces is pseudocompact.

Proof. Taking only basic open sets in the product, it is enough to consider the case of countably many factors. Suppose \( X = X_1 \times X_2 \times \cdots \) is a sequence of \( \mathcal{B}_F \)-spaces and let \( X = \prod_{n=1}^{\infty} X_n \) be its topological product. Let \( W_n = \prod_{n=1}^{\infty} U_n^{(s)} \) be a box in \( X \) with non-empty open factors \( U_n^{(s)} \subseteq X_n \) and \( X_n = U_n^{(s)} \) for almost every \( n \). We shall prove that the sequence \( \{W_n: s \in \omega\} \) cannot be discrete. Assuming it is discrete, we shall reach a contradiction. Let \( K_1 \subseteq X_1 \) be a compact set such that \( K_1 \cap U_1^{(s)} \neq \emptyset \) for every \( s \in L_1 \subseteq \omega \), with \( |L_1| = \omega \). Let \( K_2 \subseteq X_2 \) be a compact set such that \( K_2 \cap U_2^{(s)} \neq \emptyset \) for every \( s \in L_2 \subseteq L_1 \), with \( |L_2| = \omega \). Continuing this process indefinitely, for each \( j \in \omega \) we can find a compact set \( K_j \subseteq X_j \) and an infinite subset \( L_j \) of \( \omega \) such that \( K_j \cap U_j^{(s)} \neq \emptyset \) for every \( s \in L_j \). We may suppose also that \( L_1 \supseteq L_2 \supseteq \cdots \) and let \( K = \prod_{j=1}^{\infty} K_j \). The set \( K \) is compact by the Tychonoff product theorem. For each \( x \in K \), we may find a basic open box \( V_x \subseteq X \) such that \( V_x \cap W_n \neq \emptyset \) for at most one value of \( n \). Since \( K \) is compact, we may find a finite union \( V \) of the basic sets \( V_x \) such that \( V \supseteq K \) and \( V \cap W_n \neq \emptyset \) for at most finitely many indices \( n \in \omega \). The open set \( V \) may be expressed in the form:

\[
V = L \times \prod_{j=t+1}^{\infty} X_j
\]

where \( t \in \omega \) and \( L \) is an open set in \( X_1 \times X_2 \times \cdots \times X_t \), which contains \( K_1 \times K_2 \times \cdots \times K_t \). Indeed there is \( t \) such that

\[
V \supseteq \prod_{j=1}^{t} K_j \times \prod_{j=t+1}^{\infty} X_j.
\]

Hence, for \( s \in L_t \) we have

\[
V \cap W_s \supseteq \prod_{j=1}^{t} (K_j \cap U_j^{(s)}) \times \prod_{j=t+1}^{\infty} U_j^{(s)} \neq \emptyset
\]

This contradiction proves that the sequence \( W_1, W_2, \ldots \) cannot be discrete and hence \( X \) is pseudocompact.

We finish this paper with a short proof of a classic result (see [[7], Construction 2.3]):

Theorem 3.3. Every space \( X \) is homeomorphic to a closed subspace of a pseudocompact space \( Y \).

Proof. We can obviously assume that \( X \) is not pseudocompact. For every \( z \in \beta X - X \), we define \( E_z = \beta X - \{z\} \). We know \( E_z \) is locally compact and pseudocompact, and hence, each \( E_z \) is a \( \mathcal{B}_F \)-space. Taking the diagonal immersion \( \varphi \) of \( X \) into the product \( Y = \prod_{x \in X} E_z \), we know \( \varphi \) is a homeomorphism of \( X \) onto a closed subspace of \( Y \). But by (2.8) and (3.2), \( Y \) is pseudocompact.

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In [[1], example 3.4], J. L. Blasco gives an example of a $\mathcal{B}_F$-space which is not a pseudocompact $k_R$-space (see also [5]).

We finish this note stating the following question are open:

**Question 1.** Does there exist a non $\mathcal{B}_F$-space $X$ such that $X \times Y$ is pseudocompact for every pseudocompact space $Y$?

**Question 2.** Does every $\mathcal{B}_F$-space contain a dense subspace which is pseudocompact and $k_R$-space?

**Question 3.** Is there a $\mathcal{B}_F$-space $Y$ which cannot be expressed as the continuous image of a pseudocompact $k_R$-space?

**Referencias**


