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New integral inequalities for (s, m)- and (α, m)-preinvex functions

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Abstract

In this note, we give some estimate of the left hand side of generalized quadrature formula of Gauss-Jacobi in the cases where f and $|f|^{\lambda}$ for $\lambda > 1$, are (s, m)- and (α, m)-preinvex functions.

Keywords: (α, m)-preinvex function, (s, m)-preinvex function, Hölder inequality, power mean inequality.

Resumen

En esta nota, damos alguna estimación de otro caso de formula generalizada de cuadratura de Gauss-Jacobi en el caso donde f y $|f|^{\lambda}$ para $\lambda > 1$, son (s, m)- y (α, m)-preinvex functions.

Palabras claves: (α, m)-preinvex function, (s, m)-preinvex function, Hölder inequality, power mean inequality.
2011 MSC: 26D15, 26D20, 26A51.

1. Introduction

Let K be a nonempty closed subset of $\mathbb{R}$ and $\eta : K \times K \to \mathbb{R}$ be a continuous bi-function

Definition 1.1. *[20] A set $K \subseteq \mathbb{R}$ is said to be an invex with respect to the bifunction $\eta : K \times K \to \mathbb{R}$, if for all $x, y \in K$, we have*

$$x + t\eta(y, x) \in K.$$

In what follows we assume that $K \subset [0, \infty)$ be an invex set with respect to the bifunction $\eta : K \times K \to \mathbb{R}$.

Definition 1.2. *[8] A function $f : K \to \mathbb{R}$ is said to be (α, m)-preinvex with respect to η for some fixed $\alpha \in (0, 1]$, and $m \in (0, 1]$, if*

$$f(x + t\eta(y, x)) \le (1 - t^{\alpha}) f(x) + mt^{\alpha} f\left(\frac{y}{m}\right)$$

holds for all $x, y \in K$, *and* $t \in [0,1]$.

Example 1.3. *Let* $g(u) = \sqrt{u}$ *clearly* g *is* (α, m)*-preinvex function with respect to* η *where* $\alpha = m = \frac{1}{2}$ *and* $\eta(y,x) = \frac{1}{2}(y-x)$, $y, x \in (0,\infty)$

Clearly, for $y, x \in (0,\infty)$ *and* $t \in [0,1]$ *we have*

$$\begin{aligned} g(x + t\eta(y,x)) &= \left(\left(1-\frac{t}{2}\right)x + \frac{t}{2}y\right)^{\frac{1}{2}} \\ &\le \left(1-\frac{t}{2}\right)^{\frac{1}{2}} x^{\frac{1}{2}} + \left(\frac{1}{2}\right)^{\frac{1}{2}} t^{\frac{1}{2}} y^{\frac{1}{2}} \\ &\le \left(2^{-\frac{1}{2}} - \left(\frac{t}{2}\right)^{\frac{1}{2}}\right) x^{\frac{1}{2}} + \left(\frac{1}{2}\right)^{\frac{1}{2}} t^{\frac{1}{2}} y^{\frac{1}{2}} \\ &= \left(\frac{1}{2^{\frac{1}{2}}}\left(1 - t^{\frac{1}{2}}\right)\right) x^{\frac{1}{2}} + \left(\frac{1}{2}\right) t^{\frac{1}{2}} (2y)^{\frac{1}{2}} \\ &\le \left(1 - t^{\frac{1}{2}}\right) x^{\frac{1}{2}} + \left(\frac{1}{2}\right) t^{\frac{1}{2}} (2y)^{\frac{1}{2}} \\ &= \left(1 - t^{\frac{1}{2}}\right) g(x) + \left(\frac{1}{2}\right) t^{\frac{1}{2}} g\left(\frac{y}{\frac{1}{2}}\right), \end{aligned}$$

where we have used the fact that $(1-t)^n \le 2^{1-n} - t^n$ *for all* $t, n \in [0,1]$ *(see [4]).*

Thus g *is* $\left(\frac{1}{2}, \frac{1}{2}\right)$*-preinvex function with respect to* $\eta(y,x) = \frac{1}{2}(y-x)$.

Definition 1.4. *[10] A function* $f : K \subset [0, b^*] \to \mathbb{R}$ *is said to be* (s,m)*-preinvex with respect to* η *for some fixed* $s \in (0,1]$, *where* $b^* > 0$ *and* $m \in (0,1]$, *if*

$$f(x + t\eta(y,x)) \le (1-t)^s f(x) + mt^s f\left(\frac{y}{m}\right)$$

holds for all $x, y \in K$, *and* $t \in [0,1]$.

Example 1.5. *Let* $\varphi(u) = \sqrt[3]{u}$ *clearly* h *is* (s,m)*-preinvex function with respect to* η *where* $s = \frac{1}{3}, m = \sqrt{\frac{2}{3}}$ *and* $\eta(y,x) = \frac{1}{2}y - 2x$, $y, x \in (0,\infty)$

Clearly, for $y, x \in (0,\infty)$ *and* $t \in [0,1]$ *we have*

$$\begin{aligned} \varphi(x + t\eta(y,x)) &= \left((1-2t)x + \frac{t}{2}y\right)^{\frac{1}{3}} \\ &\le (1-2t)^{\frac{1}{3}} x^{\frac{1}{3}} + \left(\frac{1}{2}\right)^{\frac{1}{3}} t^{\frac{1}{3}} y^{\frac{1}{3}} \\ &\le (1-t)^{\frac{1}{3}} x^{\frac{1}{3}} + \left(\frac{1}{2}\right)^{\frac{1}{3}} t^{\frac{1}{3}} y^{\frac{1}{3}} \\ &= (1-t)^{\frac{1}{3}} x^{\frac{1}{3}} + \left(\sqrt{\frac{2}{3}}\right)^{\frac{1}{3}} \left(\frac{1}{2}\right)^{\frac{1}{3}} t^{\frac{1}{3}} \left(\frac{y}{\sqrt{\frac{2}{3}}}\right)^{\frac{1}{3}} \\ &\le (1-t)^{\frac{1}{3}} \varphi(x) + \left(\sqrt{\frac{2}{3}}\right) t^{\frac{1}{3}} \varphi\left(\frac{y}{\sqrt{\frac{2}{3}}}\right), \end{aligned}$$

where we have used the facts that $(a+b)^\lambda \le a^\lambda + b^\lambda$ *for* $a, b > 0$ *and* $0 \le \lambda \le 1$.

The generalized quadrature formula of Gauss-Jacobi type has the following form

$$\int_a^b (x-a)^p (b-x)^q f(x)\,dx = \sum_{k=0}^{m} B_{m,k} f(\gamma_k) + \mathfrak{R}_m[f], \tag{1}$$

where $B_{m,k}$ are the Christoffel coefficients, γ_k are the roots of the Jacobi polynomial of degree m, and $\mathfrak{R}_m[f]$ is the remainder term (see[19]).

In [18] Özdemir et al. gave the estimate of the left hand sides of equality (1) when the function f is quasi-convex on $[a,b] \subset \mathbb{R}^+$ with $0 \leq a < b < \infty$, as follows

$$\int_a^b (x-a)^p (b-x)^q f(x)\,dx \leq (b-a)^{p+q+1} \beta(p+1, q+1) \times \text{máx}\{f(a), f(b)\}.$$

In [9] Liu discussed the cases where certain power of the modulus of the function f is quasi-convex, and (α, m)-convex.

Ahmad [1] gave the estimates of the left hand side of the equality (1) in the cases where $|f|$ and certain power of modulus of f be P-preinvex and prequasiinvex function. Meftah [13] discussed the cases where $|f|$ and $|f|^l$ are s-preinvex functions in the second sense.

About some recent papers related to this subject, one can see [3, 5, 6, 7, 10, 11, 12, 14, 15, 16, 17].

Motivated by the results given in [1, 8, 13], in the present note we establish the estimate of the left hand side of generalized quadrature formula of Gauss-Jacobi in the cases where f and $|f|^\lambda$ for $\lambda > 1$, are (s,m)- and (α, m)-preinvex functions.

2. Main results

In order to prove the results we need the following lemma

Lemma 2.1. *[1] Let $f : S = [a, a+\eta(b,a)] \to \mathbb{R}$ be continuous function on the interval of real numbers S°(interior of S) with $a < a+\eta(b,a)$ such that $f \in L([a, a+\eta(b,a)])$, then the equality*

$$\int_a^{a+\eta(b,a)} (x-a)^p (a+\eta(b,a)-x)^q f(x)\,dx = (\eta(b,a))^{p+q+1} \int_0^1 (1-t)^q t^p f(a+t\eta(b,a))\,dt$$

holds for some fixed $p, q > 0$.

Theorem 2.2. *Let $f : [a, a+\eta(b,a)] \subset [0,\infty) \to [0,\infty)$ be integrable function on $[a, a+\eta(b,a)]$ with $\eta(b,a) > 0$. If f is (s,m)-preinvex function for some fixed $s, m \in (0,1]$ and $p, q > 0$, we have*

$$\int_a^{a+\eta(b,a)} (x-a)^p (a+\eta(b,a)-x)^q f(x)\,dx \leq (\eta(b,a))^{p+q+1} \left(f(a)\beta(p+1, q+s+1) + mf\left(\tfrac{b}{m}\right)\beta(p+s+1, q+1) \right),$$

where $\beta(.,.)$ is the Beta function.

Proof. From Lemma 2.1, and (s,m)-preinvexity of f, we have

$$\begin{aligned}
&\int_a^{a+\eta(b,a)} (x-a)^p (a+\eta(b,a)-x)^q f(x)\,dx \\
= &\ (\eta(b,a))^{p+q+1} \int_0^1 (1-t)^q t^p f(a+t\eta(b,a))\,dt \\
\leq &\ (\eta(b,a))^{p+q+1} \left(f(a) \int_0^1 t^p (1-t)^{q+s}\,dt + mf\left(\tfrac{b}{m}\right) \int_0^1 t^{p+s} (1-t)^q\,dt \right) \\
= &\ (\eta(b,a))^{p+q+1} \left(f(a)\beta(p+1,q+s+1) + mf\left(\tfrac{b}{m}\right)\beta(p+s+1,q+1) \right),
\end{aligned}$$

which is the desired results. ■

Remark 2.3. *Theorem 2.2 will be reduced to Theorem 2.2 from [13], if we take $m=1$.*

Theorem 2.4. *Let $f : [a, a+\eta(b,a)] \subset [0,\infty) \to [0,\infty)$ be integrable function on $[a, a+\eta(b,a)]$ with $\eta(b,a) > 0$ and let $\lambda > 1$. If $|f|^\lambda$ is (s,m)-preinvex function for some fixed $s, m \in (0,1]$ and $p, q > 0$, we have*

$$\begin{aligned}
&\int_a^{a+\eta(b,a)} (x-a)^p (a+\eta(b,a)-x)^q f(x)\,dx \\
\leq &\ (\eta(b,a))^{p+q+1} (\beta(p+1,q+1))^{1-\frac{1}{\lambda}} \\
&\times \left(\beta(p+1,q+s+1) |f(a)|^\lambda + m\beta(p+s+1,q+1) \left| f\left(\tfrac{b}{m}\right) \right|^\lambda \right)^{\frac{1}{\lambda}},
\end{aligned}$$

where $\beta(.,.)$ is the Beta function.

Proof. From Lemma 2.1, properties of modulus, power mean inequality, and (s, m)-preinvexity of $|f|^{\lambda}$, we have

$$
\begin{aligned}
&\int_{a}^{a+\eta(b,a)} (x-a)^{p} (a+\eta(b,a)-x)^{q} f(x)\,dx \\
\leq\;& (\eta(b,a))^{p+q+1} \left(\int_{0}^{1} (1-t)^{q} t^{p} dt \right)^{1-\frac{1}{\lambda}} \left(\int_{0}^{1} (1-t)^{q} t^{p} |f(a+t\eta(b,a))|^{\lambda} dt \right)^{\frac{1}{\lambda}} \\
\leq\;& (\eta(b,a))^{p+q+1} (\beta(p+1,q+1))^{1-\frac{1}{\lambda}} . \\
&\times \left(\int_{0}^{1} (1-t)^{q} t^{p} \left((1-t)^{s} |f(a)|^{\lambda} + m t^{s} \left| f\left(\tfrac{b}{m}\right) \right|^{\lambda} \right) dt \right)^{\frac{1}{\lambda}} \\
=\;& (\eta(b,a))^{p+q+1} (\beta(p+1,q+1))^{1-\frac{1}{\lambda}} \\
&\times \left(|f(a)|^{\lambda} \int_{0}^{1} t^{p} (1-t)^{q+s} dt + m \left| f\left(\tfrac{b}{m}\right) \right|^{\lambda} \int_{0}^{1} (1-t)^{q} t^{p+s} dt \right)^{\frac{1}{\lambda}} \\
=\;& (\eta(b,a))^{p+q+1} (\beta(p+1,q+1))^{1-\frac{1}{\lambda}} \\
&\times \left(\beta(p+1,q+s+1) |f(a)|^{\lambda} + m\beta(p+s+1,q+1) \left| f\left(\tfrac{b}{m}\right) \right|^{\lambda} \right)^{\frac{1}{\lambda}},
\end{aligned}
$$

which is the desired result. ■

Remark 2.5. *Theorem 2.4 will be reduced to Theorem 2.3 from [13], if we take* $m = 1$.

Theorem 2.6. *Suppose that all the assumptions of Theorem 2.4 are satisfied, then we have*

$$
\begin{aligned}
&\int_{a}^{a+\eta(b,a)} (x-a)^{p} (a+\eta(b,a)-x)^{q} f(x)\,dx \\
\leq\;& (\eta(b,a))^{p+q+1} \left(\beta\left(\tfrac{p\lambda}{\lambda-1}+1, \tfrac{q\lambda}{\lambda-1}+1 \right) \right)^{1-\frac{1}{\lambda}} \left(\frac{|f(a)|^{\lambda} + m \left| f\left(\frac{b}{m}\right) \right|^{\lambda}}{s+1} \right)^{\frac{1}{\lambda}},
\end{aligned}
$$

where $\beta(.,.)$ *is the Beta function.*

Proof. From Lemma 2.1, properties of modulus, Hölder inequality, and (s,m)-preinvexity of $|f|^{\lambda}$, we have

$$\begin{aligned}
&\int_{a}^{a+\eta(b,a)} (x-a)^{p}\,(a+\eta(b,a)-x)^{q}\, f(x)\,dx \\
\leq\;& (\eta(b,a))^{p+q+1}\left(\int_{0}^{1}(1-t)^{\frac{q\lambda}{\lambda-1}}\, t^{\frac{p\lambda}{\lambda-1}}\,dt\right)^{1-\frac{1}{\lambda}}\left(\int_{0}^{1}|f(a+t\eta(b,a))|^{\lambda}\,dt\right)^{\frac{1}{\lambda}} \\
\leq\;& (\eta(b,a))^{p+q+1}\left(\beta\left(\tfrac{p\lambda}{\lambda-1}+1,\tfrac{q\lambda}{\lambda-1}+1\right)\right)^{1-\frac{1}{\lambda}} \\
&\times\left(\int_{0}^{1}(1-t)^{s}|f(a)|^{\lambda}+mt^{s}\left|f\left(\tfrac{b}{m}\right)\right|^{\lambda}dt\right)^{\frac{1}{\lambda}} \\
=\;& (\eta(b,a))^{p+q+1}\left(\beta\left(\tfrac{p\lambda}{\lambda-1}+1,\tfrac{q\lambda}{\lambda-1}+1\right)\right)^{1-\frac{1}{\lambda}}\left(\frac{|f(a)|^{\lambda}+m\left|f\left(\frac{b}{m}\right)\right|^{\lambda}}{s+1}\right)^{\frac{1}{\lambda}},
\end{aligned}$$

which is the desired result. ■

Remark 2.7. *Theorem 2.6 will be reduced to Theorem 2.4 from [13], if we take $m=1$.*

Theorem 2.8. *Let $f:[a,a+\eta(b,a)]\subset[0,\infty)\to[0,\infty)$ be integrable function on $[a,a+\eta(b,a)]$ with $\eta(b,a)>0$. If f is (α,m)-preinvex function for some fixed $\alpha,m\in(0,1]$ and $p,q>0$, we have*

$$\begin{aligned}
&\int_{a}^{a+\eta(b,a)} (x-a)^{p}\,(a+\eta(b,a)-x)^{q}\, f(x)\,dx \leq (\eta(b,a))^{p+q+1} \\
&\times\Big((\beta(p+1,q+1)-\beta(p+\alpha+1,q+1))\,f(a)+\;mf\left(\tfrac{b}{m}\right)\beta(p+\alpha+1,q+1)\Big),
\end{aligned}$$

where $\beta(.,.)$ is the Beta function.

Proof. From Lemma 2.1, and (α, m)-preinvexity of $|f|^{\lambda}$,we have of f, we have

$$
\begin{aligned}
&\int_a^{a+\eta(b,a)} (x-a)^p (a+\eta(b,a)-x)^q f(x)\,dx \\
= & \ (\eta(b,a))^{p+q+1} \int_0^1 (1-t)^q t^p f(a+t\eta(b,a))\,dt \\
\leq & \ (\eta(b,a))^{p+q+1} \\
& \times \left(f(a) \left(\int_0^1 t^p (1-t)^q\,dt - \int_0^1 t^{p+\alpha} (1-t)^q\,dt \right) + m f\left(\tfrac{b}{m}\right) \int_0^1 (1-t)^q t^{p+\alpha} dt \right) \\
= & \ (\eta(b,a))^{p+q+1} \\
& \times \Big(\beta(p+1,q+1) - \beta(p+\alpha+1,q+1) f(a) + \ m f\left(\tfrac{b}{m}\right) \beta(p+\alpha+1,q+1) \Big) \\
= & \ (\eta(b,a))^{p+q+1} \\
& \times \Big(\beta(p+1,q+1) - \beta(p+\alpha+1,q+1) f(a) + \ m f\left(\tfrac{b}{m}\right) \beta(p+\alpha+1,q+1) \Big),
\end{aligned}
$$

which is the desired results. ■

Theorem 2.9. *Let $f : [a, a+\eta(b,a)] \subset [0,\infty) \to [0,\infty)$ be integrable function on $[a, a+\eta(b,a)]$ with $\eta(b,a) > 0$ and let $\lambda > 1$. If $|f|^{\lambda}$ is (α, m)-preinvex function for some fixed $\alpha, m \in (0,1]$ and $p, q > 0$, we have*

$$
\begin{aligned}
&\int_a^{a+\eta(b,a)} (x-a)^p (a+\eta(b,a)-x)^q f(x)\,dx \leq (\eta(b,a))^{p+q+1} (\beta(q+1,p+1))^{1-\frac{1}{\lambda}} \\
&\times \left((\beta(p+1,q+1) - \beta(p+\alpha+1,q+1)) |f(a)|^{\lambda} + m\beta(p+\alpha+1,q+1) \left| f\left(\tfrac{b}{m}\right) \right|^{\lambda} \right)^{\frac{1}{\lambda}},
\end{aligned}
$$

where $\beta(.,.)$ is the Beta function.

Proof. From Lemma 2.1, properties of modulus, power mean inequality, and (α, m)-preinvexity of $|f|^{\lambda}$,we have

$$\begin{aligned}
&\int_{a}^{a+\eta(b,a)} (x-a)^{p} (a+\eta(b,a)-x)^{q} f(x)\,dx \\
\leq\;& (\eta(b,a))^{p+q+1} \left(\int_{0}^{1} (1-t)^{q} t^{p} dt\right)^{1-\frac{1}{\lambda}} \left(\int_{0}^{1} (1-t)^{q} t^{p} |f(a+t\eta(b,a))|^{\lambda} dt\right)^{\frac{1}{\lambda}} \\
\leq\;& (\eta(b,a))^{p+q+1} (\beta(p+1,q+1))^{1-\frac{1}{\lambda}} \\
&\times\left(\left(\int_{0}^{1} (1-t)^{q} t^{p} dt - \int_{0}^{1} (1-t)^{q} t^{p+\alpha} dt\right)|f(a)|^{\lambda} + m\left|f\left(\tfrac{b}{m}\right)\right|^{\lambda} \int_{0}^{1} (1-t)^{q} t^{p+\alpha} dt\right)^{\frac{1}{\lambda}} \\
=\;& (\eta(b,a))^{p+q+1} (\beta(q+1,p+1))^{1-\frac{1}{\lambda}} \\
&\times\left((\beta(p+1,q+1)-\beta(p+\alpha+1,q+1))\,|f(a)|^{\lambda} + m\beta(p+\alpha+1,q+1)\left|f\left(\tfrac{b}{m}\right)\right|^{\lambda}\right)^{\frac{1}{\lambda}}
\end{aligned}$$

which is the desired result. ■

Theorem 2.10. *Suppose that all the assumptions of Theorem 2.9 are satisfied, then we have*

$$\begin{aligned}
&\int_{a}^{a+\eta(b,a)} (x-a)^{p} (a+\eta(b,a)-x)^{q} f(x)\,dx \\
\leq\;& (\eta(b,a))^{p+q+1} \left(\beta\left(\tfrac{p\lambda}{\lambda-1}+1, \tfrac{q\lambda}{\lambda-1}+1\right)\right)^{1-\frac{1}{\lambda}} \\
&\times\left(\tfrac{\alpha}{\alpha+1}|f(a)|^{\lambda} + m\tfrac{1}{\alpha+1}\left|f\left(\tfrac{a}{m}\right)\right|^{\lambda}\right)^{\frac{1}{\lambda}},
\end{aligned}$$

where $\beta(.,.)$ is the Euler beta function.

Proof. From Lemma 2.1, properties of modulus, Hölder inequality, and (α, m)-preinvexity of $|f|^{\lambda}$,we have

$$
\begin{aligned}
&\int_{a}^{a+\eta(b,a)} (x-a)^{p}\,(a+\eta(b,a)-x)^{q}\,f(x)\,dx \\
\leq\;& (\eta(b,a))^{p+q+1}\left(\int_{0}^{1}(1-t)^{\frac{q\lambda}{\lambda-1}}\,t^{\frac{p\lambda}{\lambda-1}}\,dt\right)^{1-\frac{1}{\lambda}}\left(\int_{0}^{1}|f(a+t\eta(b,a))|^{\lambda}\,dt\right)^{\frac{1}{\lambda}} \\
\leq\;& (\eta(b,a))^{p+q+1}\left(\beta\left(\tfrac{p\lambda}{\lambda-1}+1,\tfrac{q\lambda}{\lambda-1}+1\right)\right)^{1-\frac{1}{\lambda}} \\
&\times\left(\int_{0}^{1}(1-t^{\alpha})\,|f(a)|^{\lambda}+mt^{\alpha}\left|f\left(\tfrac{a}{m}\right)\right|^{\lambda}dt\right)^{\frac{1}{\lambda}} \\
=\;& (\eta(b,a))^{p+q+1}\left(\beta\left(\tfrac{p\lambda}{\lambda-1}+1,\tfrac{q\lambda}{\lambda-1}+1\right)\right)^{1-\frac{1}{\lambda}} \\
&\times\left(\tfrac{\alpha}{\alpha+1}\,|f(a)|^{\lambda}+m\tfrac{1}{\alpha+1}\left|f\left(\tfrac{a}{m}\right)\right|^{\lambda}\right)^{\frac{1}{\lambda}},
\end{aligned}
$$

which is the desired result. ∎

3. Some applications

We shall consider the following special mean

The arithmetic mean: $A(a,b)=\frac{a+b}{2}$

At first we recall the following results

Theorem 3.1. *If $f:[a,b]\to\mathbb{R}$ is convex, and $w:[a,b]\to\mathbb{R}$, $w\geq 0$, integrable and symmetric about $\frac{a+b}{2}$, then*

$$f\left(\frac{a+b}{2}\right)\int_{a}^{b}w(x)\,dx\leq\frac{1}{b-a}\int_{a}^{b}f(x)w(x)\,dx\leq\frac{f(a)+f(b)}{2}\int_{a}^{b}w(x)\,dx.$$

Theorem 3.2. *[2] Let h be defined on $[0,\text{máx}\{1,b-a\}]$ and $f:[a,b]\to\mathbb{R}$ be h-convex, $w:[a,b]\to\mathbb{R}$, $w\geq 0$, symmetric with respect to $\frac{a+b}{2}$ and $\int_{a}^{b}w(x)\,dx>0$, then*

$$\frac{f\left(\frac{a+b}{2}\right)}{2h\left(\frac{1}{2}\right)}\int_{a}^{b}w(x)\,dx\leq\frac{1}{b-a}\int_{a}^{b}f(x)w(x)\,dx.$$

Proposition 3.3. *Let $a,b\in\mathbb{R}$, with $0<a<b$, then the following inequality holds*

$$A^{\frac{1}{3}}\left(a,\frac{b-2a}{2}\right)\leq\frac{2\sqrt[3]{4}\beta\left(p+1,p+\frac{4}{3}\right)\left(\sqrt[3]{3a}+\sqrt[3]{2b}\right)}{(b-4a)\sqrt[3]{3}\beta(p+1,p+1)}.$$

Proof. Applying Theorem 2.2 for function $\varphi(u) = \sqrt[3]{u}$ given in Example 2 which is $\left(\frac{1}{3}, \sqrt{\frac{2}{3}}\right)$-preinvex with respect to $\eta(b,a) = \frac{1}{2}b - 2a$, and taking $p = q$ we have

$$\begin{aligned} &\int_a^{\frac{1}{2}b-a} (x-a)^p \left(\frac{1}{2}b - a - x\right)^p \varphi(x)\,dx \\ \leq\ & \left(\tfrac{1}{2}b - 2a\right)^{2p+1} \beta\left(p+1, p+\tfrac{4}{3}\right)\left((a^{\frac{1}{3}} + \left(\sqrt[3]{\tfrac{2}{3}}\right) b^{\frac{1}{3}}\right). \end{aligned} \tag{2}$$

Also $\varphi(u) = \sqrt[3]{u}$ is $\frac{1}{3}$-convex function in the second sense on $\left[a, \frac{1}{2}b - a\right]$ and the function $(x-a)^p\left(\frac{1}{2}b - a - x\right)^p$ is positive and symmetric about $\frac{b}{4}$ from Theorem 3.2 by taking $h(t) = t^{\frac{1}{3}}$ we obtain

$$\begin{aligned} &\left(\tfrac{1}{2}b - 2a\right)\frac{\varphi\left(\frac{a+b}{2}\right)}{2\left(\frac{1}{2}\right)^{\frac{1}{3}}} \int_a^{\frac{1}{2}b-2a} (x-a)^p\left(\tfrac{1}{2}b - a - x\right)^p dx \\ =\ & \frac{\left(\frac{1}{2}b-2a\right)^{2p+2}}{2^{\frac{2}{3}}} A^{\frac{1}{3}}\left(a, \frac{1}{2}b - a\right)\int_0^1 t^p (1-t)^p\,dt \\ =\ & \frac{\left(\frac{1}{2}b-2a\right)^{2p+2}}{2^{\frac{2}{3}}} A^{\frac{1}{3}}\left(a, \frac{1}{2}b - a\right)\beta(p+1, p+1) \\ \leq\ & \int_a^{\frac{1}{2}b-a} (x-a)^p\left(\frac{1}{2}b - a - x\right)^p \varphi(x)\,dx. \end{aligned} \tag{3}$$

From (2) and (3) we get the desired result. ■

Proposition 3.4. *Let $a, b \in \mathbb{R}$, with $0 < a < b$, then the following inequality holds*

$$A^{\frac{1}{2}}\left(a, \tfrac{a+b}{2}\right) \leq \frac{2\beta(p+1,p+1)\sqrt{2a} + 2\beta\left(p+\frac{3}{2}, p+1\right)\left(\sqrt{b} - \sqrt{2a}\right)}{(b-a)\beta(p+1,p+1)}.$$

Proof. Applying Theorem 2.8 for function $g(u) = \sqrt{u}$ given in Example 1 which is $\left(\frac{1}{2}, \frac{1}{2}\right)$-preinvex function with respect to $\eta(y,x) = \frac{1}{2}(y - x)$ with $p = q = 1$ we have

$$\begin{aligned} &\int_a^{\frac{a+b}{2}} (x-a)^p\left(\tfrac{a+b}{2} - x\right)^p g(x)\,dx \leq \left(\tfrac{b-a}{2}\right)^{2p+1} \\ &\times\left(\beta(p+1, p+1)\sqrt{a} + \beta\left(p + \frac{3}{2}, p+1\right)\left(\tfrac{\sqrt{b}}{\sqrt{2}} - \sqrt{a}\right)\right). \end{aligned} \tag{4}$$

On the other hand the function $g(u) = \sqrt{u}$ is $\frac{1}{2}$-convex in the second sense on $\left[a, \frac{a+b}{2}\right]$ and the function

$(x-a)^p\left(\frac{a+b}{2}-x\right)^p$ is positive and symmetric about $\frac{3a+b}{4}$ from Theorem 3.2 by taking $h(t)=t^{\frac{1}{2}}$ we obtain

$$\begin{aligned}
&\frac{b-a}{2}\frac{g\left(\frac{3a+b}{4}\right)}{2\left(\frac{1}{2}\right)^{\frac{1}{2}}}\int\limits_a^{\frac{a+b}{2}}(x-a)^p\left(\frac{a+b}{2}-x\right)^p dx \\
=\ & \left(\frac{b-a}{2}\right)^{2p+2}\frac{A^{\frac{1}{2}}\left(a,\frac{a+b}{2}\right)}{\sqrt{2}}\beta(p+1,p+1) \\
\leq\ & \int\limits_a^{\frac{a+b}{2}}(x-a)^p\left(\frac{a+b}{2}-x\right)^p g(x)\,dx. \qquad (5)
\end{aligned}$$

From (4) and (5) we get the desired result. ■

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